August 31 Math 1190 sec. 51 Fall 2016

Section 1.3: Continuity

Compositions

Suppose $\lim_{x\to c} g(x) = L,$ and f is continuous at L, then

$$\lim_{x\to c} f(g(x)) = f(L) \quad \text{i.e.} \quad \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right).$$

Theorem: If g is continuous at c and f is continuous at $g(c)$, then $(f \circ g)(x)$ is continuous at c.

Essentially, this says that "compositions of continuous functions are continuous."

Example

Suppose we know that $f(x) = e^x$ is continuous on $(-\infty, \infty)^*$. Evaluate

$$\lim_{x \to \sqrt{\ln(3)}} e^{x^2 + \ln(2)}$$

If $g(x) = x^2 + \ln 2$, then

g is continuous @ $\sqrt{\ln 3}$

If $f(x) = e^x$ then

$$e^{x^2 + \ln 2} = f(g(x))$$

So

$$\lim_{x \to \sqrt{\ln 3}} e^{x^2 + \ln 2}$$

$$= e^{(\sqrt{\ln 3})^2 + \ln 2}$$

$$= e^{\ln 3 + \ln 2} = e^{\ln 3} \cdot e^{\ln 2} = 3 \cdot 2 = 6$$

$$e^{a+b} = e^a \cdot e^b$$

*This is true.

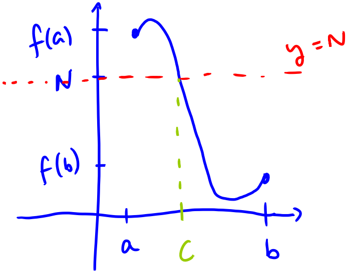
Inverse Functions

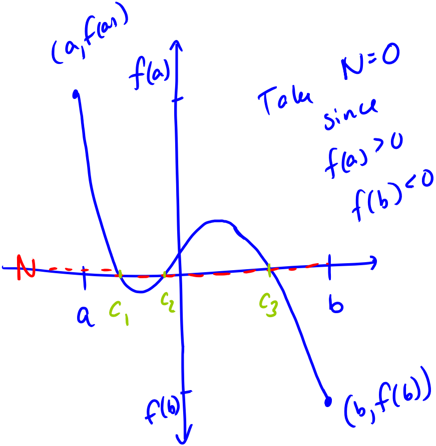
Theorem: If f is a one to one function that is continuous on its domain, then its inverse function f^{-1} is continuous on its domain.

Continuous functions (with inverses) have continuous inverses.

Theorem:

Intermediate Value Theorem (IVT) Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists c in the interval (a, b) such that $f(c) = N$.





Application of the IVT

Show that the equation has at least one solution in the interval.

$$\sqrt{x-1} = 4 - x^2 \quad 1 \leq x \leq 2$$

Let $f(x) = \sqrt{x-1} - 4 + x^2$

Observations: ① If $f(c) = 0$ then $\sqrt{c-1} - 4 + c^2 = 0$

i.e. $\sqrt{c-1} = 4 - c^2$

c would solve our equation.

② f is the sum of continuous functions.

Hence it is continuous provided $x - 1 \geq 0$

i.e. $x \geq 1$.

Thus f is continuous on $[1, 2]$.

$f(x) = \sqrt{x-1} - 4 + x^2$ here $a = 1$ $b = 2$

$f(1) = \sqrt{1-1} - 4 + 1^2 = -3$ $f(a) < 0$

$f(2) = \sqrt{2-1} - 4 + 2^2 = 1$ $f(b) > 0$

The number $N = 0$ is between $f(1)$ and $f(2)$,

By the IVT, there exists some c in $(1, 2)$

such that $f(c) = 0$.

This doesn't tell us what the root c is, but it does guarantee that our equation $\sqrt{x-1} = 4 - x^2$ has at least one solution c in the interval $[1, 2]$.

Section 1.4: Limits and Continuity of Trigonometric, Exponential and Logarithmic Functions

Here we list without proof[†] the continuity properties of several well known functions.

$\sin x$: The sine function $y = \sin x$ is continuous on its domain $(-\infty, \infty)$.

$\cos x$: The cosine function $y = \cos x$ is continuous on its domain $(-\infty, \infty)$.

e^x: The exponential function $y = e^x$ is continuous on its domain $(-\infty, \infty)$.

$\ln(x)$: The natural log function $y = \ln(x)$ is continuous on its domain $(0, \infty)$.

[†] You are already familiar with their graphs.

Additional Functions

- By the quotient property, each of $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are continuous on each of their respective domains.

- For $a > 0$ with $a \neq 1$, the function

  $$a^x = e^{x \ln a}.$$

  $e^{x \ln a} = e^{\ln a^x} = a^x$

  By the composition property, each exponential function $y = a^x$ is continuous on $(-\infty, \infty)$.

- For $a > 0$ with $a \neq 1$, the function

  $$\log_a(x) = \frac{\ln x}{\ln a}.$$

  By the constant multiple property, each logarithm function $y = \log_a(x)$ is continuous on $(0, \infty)$.

Example

Evaluate each limit.

(a) $\lim_{x \to \pi} \cos(x + \sin x)$ $= \cos(\pi + \sin\pi)$

$= \cos(\pi + 0) = \cos\pi = -1$

Composition of continuous functions

(b) $\lim_{t \to \frac{\pi}{4}} e^{\tan t}$ $= e^{\tan\frac{\pi}{4}} = e^{1} = e$

$\frac{\pi}{4}$ is in the domain of $f(t) = \tan t$.

Question

Evaluate the limit $\lim_{x \to \pi} \ln(\cos^2 x)$.

(a) e

(b) 1

(c) DNE

(d) 0

$\lim_{x \to \pi} \ln(\cos^2 x) = \ln(\cos^2 \pi)$

$= \ln((-1)^2)$

$= \ln 1 = 0$

$e^{\boxed{0}} = 1$

Squeeze Theorem:

Theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for all x in an interval containing c except possibly at c. If

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} g(x) = L.$$

Squeeze Theorem:

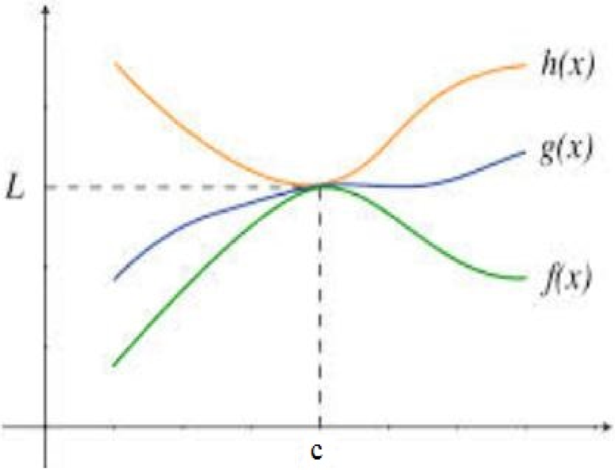


Figure: Graphical Representation of the Squeeze Theorem.

Example: Evaluate

$$\lim_{\theta \to 0} \theta^2 \sin \frac{1}{\theta}$$

For all θ, note

$$-1 \leq \sin\frac{1}{\theta} \leq 1$$

since $\theta^2 > 0$ $\quad -\theta^2 \leq \theta^2 \sin\frac{1}{\theta} \leq \theta^2$

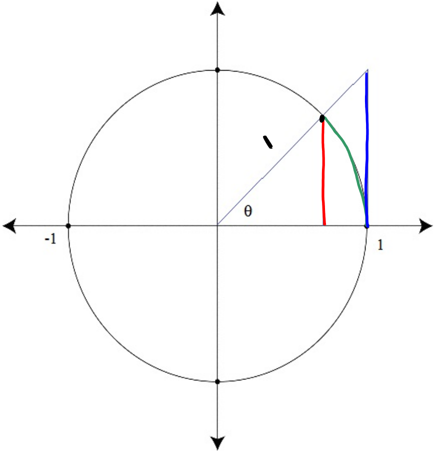
$\lim_{\theta \to 0} -\theta^2 = -0^2 = 0$ and $\lim_{\theta \to 0} \theta^2 = 0^2 = 0$

By the squeeze thm $\lim_{\theta \to 0} \theta^2 \sin\frac{1}{\theta} = 0$ as well.

Here $f(\theta) = -\theta^2$, $h(\theta) = \theta^2$ and $g(\theta) = \theta^2 \sin\frac{1}{\theta}$

A Couple of Important Limits

Theorem: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$



Lengths

— $\sin\theta$ $\left(\frac{opp}{hyp} = \frac{opp}{1}\right)$

— θ $(s = r\theta = 1\theta)$

— $\tan\theta$ $\left(\frac{opp}{adj} = \frac{opp}{1}\right)$

$\sin\theta \le \theta \le \tan\theta$

for small θ (near zero)

Inequality 1 : $\sin\theta \le \theta \Rightarrow \dfrac{\sin\theta}{\theta} \le 1$

for $\theta > 0$

Since $\sin\theta$ is odd

we have $\dfrac{\sin\theta}{\theta} \le 1$ for all small θ.

Inequality 2 : $\theta \le \tan\theta \Rightarrow \theta \le \dfrac{\sin\theta}{\cos\theta}$

for $\theta > 0$ $\cos\theta \le \dfrac{\sin\theta}{\theta}$

Since $\sin\theta$ is odd and $\cos\theta$ is even

$\cos\theta \le \dfrac{\sin\theta}{\theta}$ for all small θ.

We have $\quad \cos\theta \le \dfrac{\sin\theta}{\theta} \le 1$

$\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos\theta = \cos 0 = 1$

By the squeeze thm

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

The statement $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$ is offered without proof.

Important Observation

The character used in the limit statement is immaterial. That is,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{t\to 0}\frac{\sin t}{t} = \lim_{\heartsuit\to 0}\frac{\sin(3\heartsuit)}{3\heartsuit} = 1$$

The key is that the argument of the sine matches the denominator with these tending to zero.

This is a limit. It should not be confused with the statement

$$\text{,,}\ \frac{\sin\theta}{\theta} = 1\text{''}$$

which is NEVER true.

Examples

Evaluate each limit if possible.

(a) $\displaystyle \lim_{x\to 0} \frac{\sin(4x)}{x}$

Note

$$\lim_{x\to 0} \frac{\sin(4x)}{4x} = 1$$

$$= \lim_{x\to 0} \frac{\sin(4x)}{x} \cdot \frac{4}{4}$$

$$= \lim_{x\to 0} 4 \frac{\sin(4x)}{4x}$$

$$= 4 \lim_{x\to 0} \frac{\sin(4x)}{4x} = 4 \cdot 1 = 4$$

Note: the $4x$ inside the sine is out of our control

(b) $\lim_{t\to 0} \dfrac{2t}{\tan(3t)} = \lim_{t\to 0} \dfrac{2t}{\dfrac{\sin(3t)}{\cos(3t)}}$

$$= \lim_{t\to 0} \cos 3t \left(\frac{2t}{\sin 3t}\right)$$

$$= \lim_{t\to 0} 2\cos 3t \left(\frac{t}{\sin 3t}\right)\cdot\frac{3}{3}$$

$$= \lim_{t\to 0} \frac{2}{3}\cos 3t \left(\frac{3t}{\sin 3t}\right)$$

$$= \frac{2}{3}\lim_{t\to 0} \cos 3t \left(\frac{3t}{\sin 3t}\right)$$

$$= \frac{2}{3} \cos(3 \cdot 0) \frac{1}{1} = \frac{2}{3} \cdot 1 \cdot 1 = \frac{2}{3}$$