August 31 Math 2306 sec 51 Fall 2015

Section 2.3: First Order Linear Equations

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Solve the ODE

$$x^2\frac{dy}{dx}+2xy = e^x$$

Standard form

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{e^x}{x^2}$$

$$P(x) = \frac{2}{x} \qquad \int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2$$

Integrating factor $\mu(x) = e^{\int P(x)dx} = e^{\ln x^2} = x^2$

Multiply the equation in standard form by μ

$$\frac{d}{dx}\left[x^2 y\right] = \frac{e^x}{x^2}\cdot x^2$$

$$\frac{d}{dx}\left[x^2 y\right] = e^x \quad \Rightarrow$$

$$\int \frac{d}{dx}\left[x^2 y\right] dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

$$\boxed{y = \frac{e^x}{x^2} + \frac{C}{x^2}}$$

Equation in standard form

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{e^x}{x^2}$$

$\mu(x) = x^2 \quad \Rightarrow$ multiply by μ

$$x^2\frac{dy}{dx} + x^2 \cdot \frac{2}{x}y = x^2\frac{e^x}{x^2}$$

$$x^2\frac{dy}{dx} + 2xy = e^x$$

Solve the ODE

$$\frac{dy}{dx}+y=3xe^{-x}$$

Already in standard form

$P(x)=1$ $\qquad$ $\int P(x)\,dx=\int dx = x$

$\mu = e^{\int P(x)\,dx} = e^{x}$ $\qquad$ multiply eqn by μ

$$e^{x}\frac{dy}{dx}+e^{x}y=e^{x}(3xe^{-x})$$

$$\frac{d}{dx}\left[e^{x}y\right]=3x$$

$$\int \frac{d}{dx}\left[e^x y\right] dx = \int 3x \, dx$$

$$e^x y = 3\frac{x^2}{2} + C$$

$$y = \frac{\frac{3}{2}x^2}{e^x} + \frac{C}{e^x}$$

$$\boxed{y = \frac{3x^2 e^{-x}}{2} + Ce^{-x}}$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \quad y(1) = 5$$

Standard form $\quad \frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x}$

$P(x) = \frac{-1}{x} \qquad \int P(x)\,dx = \int \frac{-1}{x}\,dx = -\ln x = \ln x^{-1}$

$\mu = e^{\int P(x)dx} = e^{\ln x^{-1}} = x^{-1}$

$x^{-1}\frac{dy}{dx} - \frac{1}{x^2}y = 2x \cdot x^{-1}$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = 2x^2 + Cx$$

one parameter family of solns to the DE

$$y = 2x^2 + Cx \quad \text{and} \quad y(1) = 5$$

$$y(1) = 2(1)^2 + C \cdot 1 = 5$$

$$2 + C = 5 \implies C = 3$$

$$\boxed{\begin{array}{c}\text{The solution to the IVP is} \\ y = 2x^2 + 3x\end{array}}$$

Solve the IVP

$$\frac{dy}{dt}+\frac{4}{t}y=\frac{e^t}{t^3},\quad y(-1)=0$$

Already in standard form

$$P(t)=\frac{4}{t}$$

$$\int P(t)\,dt=\int \frac{4}{t}\,dt=4\ln|t|=\ln t^4$$

$$\mu=e^{\int P(t)\,dt}=e^{\ln t^4}=t^4$$

$$t^4\frac{dy}{dt}+t^4\cdot\frac{4}{t}y=t^4\frac{e^t}{t^3}$$

$$\frac{d}{dt}\left[t^4 y\right] = te^t$$

Int by parts

$$\int \frac{d}{dt}\left[t^4 y\right] dt = \int te^t \, dt$$

$u = t \qquad du = dt$

$v = e^t \qquad dv = e^t \, dt$

$$t^4 y = te^t - \int e^t \, dt$$

$$= te^t - e^t + C$$

$$y = \frac{te^t - e^t + C}{t^4}$$

$$y = \frac{te^t - e^t + C}{t^4}, \quad y(-1) = 0$$

$$y(-1) = \frac{-1e^{-1} - e^{-1} + C}{(-1)^4} = 0 \Rightarrow -2e^{-1} + C = 0$$

$$\Rightarrow C = 2e^{-1}$$

The solution to the IVP is

$$y = \frac{te^t - e^t + 2e^{-1}}{t^4}.$$

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**

Section 3.1 (1.3, and a peek at 3.2) Applications

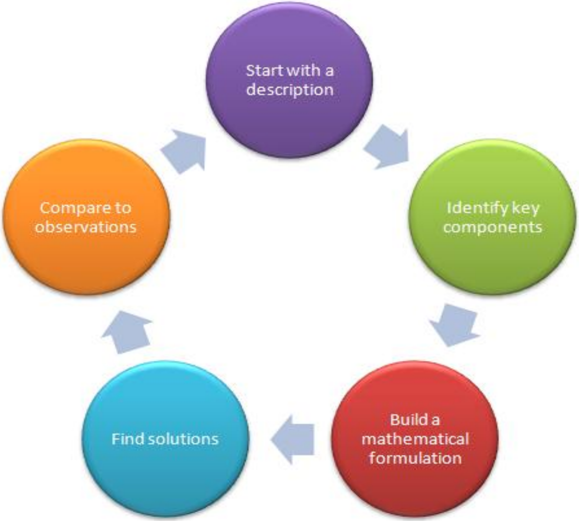

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let $P(t)$ be the population of rabbits

at time t in years.

The rate of change of P is $\frac{dP}{dt}$.

To be proportional to the current population

means to be kP for some constant k.

So $\frac{dP}{dt} = kP$ Letting 2011 correspond to $t=0$, we have

$P(0) = 58$ and $P(1) = 89$.

We have an IVP plus an additional condition. The value of k isn't known (yet).

We'll try to solve this problem next time.