August 31 Math 2306 sec. 53 Fall 2018

Section 4: First Order Equations: Exact Equations

We considered first order equations of the form

$$M(x, y) dx + N(x, y) dy = 0.$$
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The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (1) is called an **exact equation** on some rectangle R if there exists a function F(x, y) such that

$$\frac{\partial F}{\partial x} = M(x, y)$$
 and $\frac{\partial F}{\partial y} = N(x, y)$

for every (x, y) in R.

Exact Equation Solution

If M(x, y) dx + N(x, y) dy = 0 happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y)=C$$

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Recognizing Exactness

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If it is true that

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N \qquad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y}$$
this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

 $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial x}$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx+N(x,y)\,dy=0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

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Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^{2} x) dx + (x^{2} + 2y) dy = 0$$
Here $M(x,y) = 2xy - \sec^{2} x$ and $N(x,y) = x^{2} + 2y$
(here : is $\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$?
 $\frac{\partial M}{\partial y} = 2x(1) - 0 = 2x$ $\frac{\partial N}{\partial x} = 2x + 0$
 $\frac{\partial m}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow Equation is exact /$
Solutions will be given by $F(x,y) = C$
where $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$
 $= 100 \text{ M}(x,y)$

We have
$$\frac{\partial F}{\partial x} = 2xy - Se^2 x$$
 and $\frac{\partial F}{\partial y} = x^2 + 2y$
To find F we can integrate $\frac{\partial F}{\partial x}$ with respect to x.
 $F(x,y) = \int \frac{\partial F}{\partial x} dx = \int (2xy - Se^2 x) dx$
 $= x^2 y - \tan x + g(y)$
* The "constant" of integration $g(y)$ can depend on
 y Since $\frac{\partial}{\partial x} g(y)$ would be zero.
Now, we know $F(x,y) = x^2 y - \tan x + g(y)$ *

and
$$\frac{\partial F}{\partial y} = x^2 + 2y$$

From $\frac{\partial F}{\partial y} = x^2 - 0 + g'(y) = x^2 + 2y$
So matching gives
 $g'(y) = 2y$
cn antidenizative is $g(y) = y^2$
So $F(x, y) = x^2y - \tan x + y^2$

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Solutions to the ODE are given by

 $\chi^2 y - \tan x + y^2 = C$

Special Integrating Factors

Suppose that the equation M dx + N dy = 0 is not exact. Clearly our approach to exact equations would be fruitless as there is no such function F to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y-6x)\,dx+(3x-4x^2y^{-1})\,dy=0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$

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Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x, y) = xy^2$.

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Special Integrating Factors

The function μ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for μ of a certain *form* (usually $\mu = x^n y^m$ for some powers *n* and *m*). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on *x*, or there is an integrating factor $\mu = \mu(y)$ depending only on *y*.

Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M dx + N dy = 0$$

is NOT exact, but that

$$\mu M \, dx + \mu N \, dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on *y*. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}$$

Let's use the product rule in the right side.

Special Integrating Factor $\mu = \mu(x)$

Note
$$\frac{\partial}{\partial y}\mu(x)=0$$

product rule
on the
risht

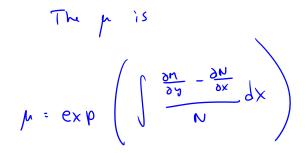
$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

$$\mu(N) \frac{\partial M}{\partial y} = \frac{d}{dx} N + \mu \frac{\partial N}{\partial x}$$
$$= \mu \left(\frac{\partial n}{\partial y} - \frac{\partial N}{\partial x} \right)$$

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$$\frac{d\mu}{dx} = \mu \left(\frac{\frac{\partial n}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) \quad \text{for } N \neq 0$$

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Special Integrating Factor

$$M\,dx + N\,dy = 0 \tag{2}$$

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Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then **...**

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \, dx\right)$$

is an special integrating factor for (2). If $(\partial N/\partial x - \partial M/\partial y)/M$ is

continuous and depends only on v, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \, dy\right)$$

is an special integrating factor for (2).

Example

Solve the equation $2xy dx + (y^2 - 3x^2) dy = 0$. M = 2xy $N = y' - 3x^2$ is it exact? $\frac{\partial M}{\partial y} = 2x$ $\frac{\partial N}{\partial x} = -6x$ not equal, not exact. 3M - 3N depend only on x? NO Does DN - Dr depend only on y? yes! 60 Poes

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= -8x 2x3 -6x - 2x 2xy 00 Depe צס ÷ 3 ٥× M

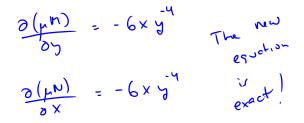
$$\mu = \mu(y) = \exp\left(\int \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dy\right)$$
$$= \int \frac{1}{2} \frac{\partial dy}{\partial y} = \frac{1}{2} \frac{\partial ny}{\partial y} = \frac{\partial ny}{\partial y}$$

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The new equation is

 $5^{4}\left(2xy dx + (y^{2} - 3x^{2}) dy\right) = 0$

 $\partial_{x} y^{-3} dx + (y^{-2} - 3x^{2} y^{4}) dy = 0$



We ran out of time, but here is the rest of the solution process. The solutions will be given by F(x, y) = C where 2F = mm = 2x 5 and $\frac{\partial F}{\partial y} = \mu N = y^2 - 3x^2 y^4$ $F(x,y) = \int \frac{\partial F}{\partial x} dx = \int 2x y^3 dx$ ◆□▶ ◆□▶ ◆ □ ▶ ◆ □ ▶ ◆ □ ●

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= x2 -3 + g(5) $\frac{\partial F}{\partial y} = -3x^{2}y^{4} + g(y)$ They Compare this to <u> 2</u> = <u>-</u>² - <u>3</u>×² - 2 16 must be that みいショー タ(ショーカ) $S_{0} = F(x,y) = x^{2}y^{3} - y^{-1}$ <ロト < 回 > < 回 > < 三 > < 三 > 三 三

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 $x^{2}y^{-3} - y^{-1} = C$

Simplified

 $\frac{x^2}{3} - \frac{1}{3} = C$

or better yet

 $\frac{x^2 - y^2}{y^3} = C$

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