August 31 Math 2306 sec. 53 Fall 2018

Section 4: First Order Equations: Exact Equations

We considered first order equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{1}$$

The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (1) is called an **exact equation** on some rectangle R if there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

for every (x,y) in R.

Exact Equation Solution

If $M(x, y)\,dx + N(x, y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x, y) = C$$

Recognizing Exactness

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

$$\frac{\partial^2 F}{\partial y\, \partial x} = \frac{\partial M}{\partial y}$$

and

$$\frac{\partial^2 F}{\partial x\, \partial y} = \frac{\partial N}{\partial x}$$

this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

Here $M(x,y) = 2xy - \sec^2 x$ and $N(x,y) = x^2 + 2y$

Check: is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?

$\frac{\partial M}{\partial y} = 2x(1) - 0 = 2x$ $\qquad$ $\frac{\partial N}{\partial x} = 2x + 0$

$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow$ Equation is exact!

Solutions will be given by $F(x,y) = C$

where $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$

We have $\dfrac{\partial F}{\partial x} = 2xy - \sec^2 x$ and $\dfrac{\partial F}{\partial y} = x^2 + 2y$

To find F we can integrate $\dfrac{\partial F}{\partial x}$ with respect to x.

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int \left(2xy - \sec^2 x\right) dx$$

$$= x^2 y - \tan x + g(y)$$

* The "constant" of integration $g(y)$ can depend on y since $\dfrac{\partial}{\partial x} g(y)$ would be zero.

Now, we know $F(x,y) = x^2 y - \tan x + g(y)$ *

and $\dfrac{\partial F}{\partial y} = x^2 + 2y$

From $*$ $\dfrac{\partial F}{\partial y} = x^2 - 0 + g'(y) = x^2 + 2y$

So matching gives

$$g'(y) = 2y$$

an antiderivative is $g(y) = y^2$

So $F(x,y) = x^2y - \tan x + y^2$

Solutions to the ODE are given by

$$x^2y - \tan x + y^2 = C$$

Special Integrating Factors

Suppose that the equation $M\,dx + N\,dy = 0$ is not exact. Clearly our approach to exact equations would be fruitless as there is no such function F to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x,y) = xy^2$.

$$xy^2(2y-6x)\,dx + xy^2(3x-4x^2y^{-1})\,dy = 0, \quad \Longrightarrow$$

$$\underbrace{(2xy^3-6x^2y^2)}_{\mu M}\,dx + \underbrace{(3x^2y^2-4x^3y)}_{\mu N}\,dy = 0$$

$$\frac{\partial(\mu M)}{\partial y} = 6xy^2 - 12x^2y$$

$$\frac{\partial(\mu N)}{\partial x} = 6xy^2 - 12x^2y$$

match!

The new equation is exact.

Special Integrating Factors

The function μ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for μ of a certain *form* (usually $\mu = x^n y^m$ for some powers n and m). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on x, or there is an integrating factor $\mu = \mu(y)$ depending only on y.

Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M\,dx + N\,dy = 0$$

is NOT exact, but that

$$\mu M\,dx + \mu N\,dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on y. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

Let's use the product rule in the right side.

Special Integrating Factor $\mu = \mu(x)$

Not $\frac{\partial}{\partial y}\mu(x) = 0$

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

product rule on the right

$$\mu(x)\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \frac{d\mu}{dx}N = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$

$$= \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\frac{d\mu}{dx} = \mu\left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) \quad \text{for } N \neq 0$$

Such μ existing requires

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad \text{depends \underline{only} on } x$$

If this test is satisfied, then μ can be found by separation of variables.

The μ is

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx \right)$$

Special Integrating Factor

$$M\,dx + N\,dy = 0 \tag{2}$$

Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

is an special integrating factor for (2). If $(\partial N/\partial x - \partial M/\partial y)/M$ is

continuous and depends only on y, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

is an special integrating factor for (2).

Example

Solve the equation $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

$M = 2xy$ $\quad N = y^2 - 3x^2$ $\quad$ is it exact?

$\dfrac{\partial M}{\partial y} = 2x$ $\quad \dfrac{\partial N}{\partial x} = -6x$ $\quad$ not equal, not exact!

Does $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ depend only on x? NO

or

Does $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ depend only on y? yes!

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x - (-6x)}{y^2 - 3x^2} = \frac{8x}{y^2 - 3x^2}$$

Depends on x and y!

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x - 2x}{2xy} = \frac{-8x}{2xy} = \frac{-4}{y}$$

Depends only on y!

$$\mu = \mu(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\, dy\right)$$

$$= e^{\int \frac{-4}{y}\, dy} = e^{-4\ln y} = e^{\ln y^{-4}} = y^{-4}$$

The new equation is

$$y^{-4}\left(2xy\,dx + (y^2 - 3x^2)\,dy\right) = 0$$

$$2xy^{-3}\,dx + (y^{-2} - 3x^2y^{-4})\,dy = 0$$

$$\frac{\partial(\mu M)}{\partial y} = -6xy^{-4}$$

$$\frac{\partial(\mu N)}{\partial x} = -6xy^{-4}$$

The new equation is exact!

We ran out of time, but here is the rest of the solution process.

The solutions will be given by

$F(x,y) = C$ where

$$\frac{\partial F}{\partial x} = \mu M = 2x y^{-3} \quad \text{and}$$

$$\frac{\partial F}{\partial y} = \mu N = y^{-2} - 3x^2 y^{-4}$$

$$F(x,y) = \int \frac{\partial F}{\partial x}\, dx = \int 2x y^{-3}\, dx$$

$$= x^2 y^{-3} + g(y)$$

Then $\dfrac{\partial F}{\partial y} = -3x^2 y^{-4} + g'(y)$

Compare this to

$$\frac{\partial F}{\partial y} = y^{-2} - 3x^2 y^{-4}$$

It must be that

$$g'(y) = y^{-2} \implies g(y) = -y^{-1}$$

So $F(x,y) = x^2 y^{-3} - y^{-1}$

Solutions are

$$x^2 y^{-3} - y^{-1} = C$$

Simplified

$$\frac{x^2}{y^3} - \frac{1}{y} = C$$

or better yet

$$\frac{x^2 - y^2}{y^3} = C$$