August 31 Math 2306 sec 54 Fall 2015

Section 2.3: First Order Linear Equations

We were trying to solve the first order linear equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x).$$

Recall that the solution will look like $y = y_c + y_p$ where the *complementary solution* y_c solves the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0,$$

and the *particular solution* y_p depends on $f(x)$.

$\frac{dy}{dx} + P(x)y = f(x)$

We sought a function, called an *integrating factor*, $\mu(x)$ such that when we multiply our equation through by μ, the left hand side would become

$$\frac{d}{dx}\left[\mu(x)y(x)\right].$$

We solved this intermediate problem and came up with

$$\mu(x) = \exp\left(\int P(x)\,dx\right).$$

Multiply the D.E. by μ

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x)$$

$$\frac{d}{dx}\left[\mu(x) y\right] = \mu(x) f(x)$$

Check: $\frac{d}{dx}\left[\mu(x) y\right] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$

$$\frac{d\mu}{dx} = \exp\left(\int P(x)\,dx\right) \cdot P(x) = \mu P$$

$$\int \frac{d}{dx}\left[\mu(x) y\right] dx = \int \mu(x) f(x)\, dx$$

$$\mu(x)\, y(x) = \int \mu(x) f(x)\, dx + C$$

$$\Rightarrow \quad y(x) = \underbrace{\frac{1}{\mu(x)} \int \mu(x) f(x)\, dx}_{y_p} + \underbrace{\frac{C}{\mu(x)}}_{y_c}$$

$$y(x) = e^{-\int P(x)dx} \int \left(e^{\int P(t)dt} \right) f(x)\, dx + C\, e^{-\int P(x)dx}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solve the ODE

Put in Standard form (assume $x \neq 0$)

$$x^2\frac{dy}{dx}+2xy = e^x$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{e^x}{x^2}$$

$$P(x) = \frac{2}{x} \qquad \int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2$$

Integrating factor $\mu(x) = e^{\int P(x)dx} = e^{\ln x^2} = x^2$

Mult. eqn in Standard form by μ

$$x^2\left(\frac{dy}{dx} + \frac{2}{x}y\right) = x^2\,\frac{e^x}{x^2}$$

$$x^2\frac{dy}{dx} + 2xy = e^x$$

$$\frac{d}{dx}\left[x^2 y\right] = e^x$$

$$\int \frac{d}{dx}\left[x^2 y\right] dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

$$y = \underbrace{\frac{e^x}{x^2}}_{y_p} + \underbrace{\frac{C}{x^2}}_{y_c}$$

Solve the ODE

$$\frac{dy}{dx} + y = 3xe^{-x}$$

Already in Standard form

$P(x) = 1 \qquad \int P(x)\,dx = \int dx = x$

$\mu(x) = e^{\int P(x)\,dx} = e^{x}$ Mult. eqn by μ

$$e^{x}\left(\frac{dy}{dx} + y\right) = e^{x}\left(3xe^{-x}\right)$$

$$e^{x}\frac{dy}{dx} + e^{x}y = 3x$$

$$\frac{d}{dx}\left[e^{x} y\right] = 3x$$

$$\int \frac{d}{dx}\left[e^{x} y\right] dx = \int 3x \, dx$$

$$e^{x} y = \frac{3x^2}{2} + C$$

$$y = \frac{1}{e^{x}}\left(\frac{3x^2}{2} + C\right)$$

$$y = \frac{3}{2} x^2 e^{-x} + C e^{-x}$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \quad y(1) = 5$$

lets assume $x > 0$

(so $x = 1$ is in our interval)

Standard form

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x$$

$$P(x) = \frac{-1}{x}$$

$$\int P(x)\,dx = \int \frac{-1}{x}\,dx = -\ln|x| = \ln x^{-1} \quad (\text{for } x>0)$$

$$\mu = e^{\int P(x)\,dx} = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = x^{-1}(2x)$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2$$

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = 2$$

$$\int \frac{d}{dx}\left[\frac{1}{x}y\right] dx = \int 2\,dx$$

$$\frac{1}{x}y = 2x + C \implies$$

one parameter family of solutions to the DE

$$y = 2x^2 + Cx$$

$$y = 2x^2 + Cx \qquad y(1) = 5$$

$$y(1) = 2(1^2) + C(1) = 5$$

$$2 + C = 5 \implies C = 3$$

$$\boxed{\text{The solution to the IVP is } \quad y = 2x^2 + 3x.}$$

Solve the IVP

$$\frac{dy}{dt} + \frac{4}{t}y = \frac{e^t}{t^3}, \quad y(-1) = 0$$

Since $t \neq 0$, we'll assume $-\infty < t < 0$.

The equation is in standard form $P(t) = \frac{4}{t}$

$$\int P(t)\,dt = \int \frac{4}{t}\,dt$$

$$= 4\ln|t| = \ln t^4$$

$$\mu = e^{\int P(t)\,dt} = e^{\ln t^4} = t^4$$

$$t^4\left(\frac{dy}{dt} + \frac{4}{t}y\right) = t^4\left(\frac{e^t}{t^3}\right)$$

$$\frac{d}{dt}\left[t^4 y\right] = te^t$$

int. by parts

$$\int \frac{d}{dt}\left[t^4 y\right] dt = \int te^t\, dt$$

$u = t \quad du = dt$

$v = e^t \quad dv = e^t dt$

$$t^4 y = te^t - \int e^t\, dt$$

$$t^4 y = te^t - e^t + C$$

$$y = \frac{te^t - e^t + C}{t^4}$$

$$y = \frac{te^t - e^t + C}{t^4}, \quad y(-1) = 0$$

$$y(-1) = \frac{-1\,e^{-1} - e^{-1} + C}{(-1)^4} = 0 \Rightarrow$$

$$-e^{-1} - e^{-1} + C = 0 \Rightarrow \quad C = 2e^{-1}$$

The solution to the IVP is

$$\boxed{y = \frac{te^t - e^t + 2e^{-1}}{t^4}.}$$

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**

Section 3.1 (1.3, and a peek at 3.2) Applications

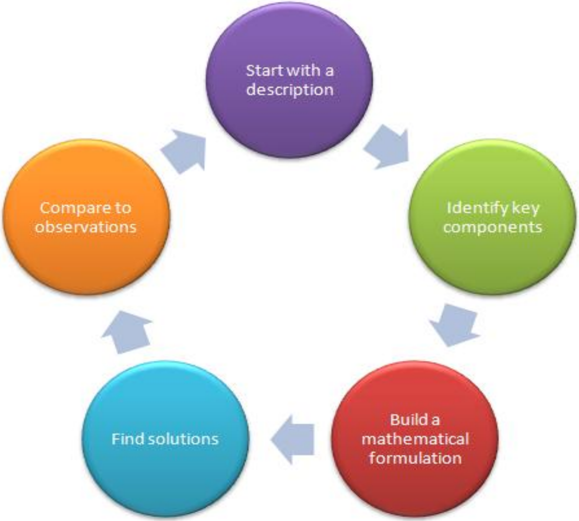

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let the population of rabbits at time t be $P(t)$ where t is in years.

The rate of change of population is $\frac{dP}{dt}$

Proportional to the current population would be kP for some constant k.

we have

$$\frac{dP}{dt} = kP$$

If we take $t=0$ in 2011

$P(0) = 58$ and $P(1) = 89$

We'll solve this problem next time.
We have an IVP plus an extra condition. Finding the value of k will be part of the problem solving process.