August 31 Math 2306 sec. 56 Fall 2017

Section 4: First Order Equations: Exact Equations

We considered first order equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{1}$$

The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (1) is called an **exact equation** on some rectangle R if there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

for every (x,y) in R.

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

A solution set to an exact equation is a relation of the form $F(x, y) = C$ for constant C where F is the function satisfying

$$\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y).$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(e^y - \sin x)\,dx + \left(xe^y + \frac{1}{1+y^2}\right)\,dy = 0$$

$M(x,y) = e^y - \sin x$ $\qquad$ $N(x,y) = xe^y + \dfrac{1}{1+y^2}$

$\dfrac{\partial M}{\partial y} = e^y - 0 = e^y$ $\qquad$ $\dfrac{\partial N}{\partial x} = 1 \cdot e^y + 0 = e^y$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ the equation is exact

Solutions are the relation $F(x,y) = C$

where $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$

$$\frac{\partial F}{\partial x} = e^y - \sin x \quad , \quad \frac{\partial F}{\partial y} = xe^y + \frac{1}{1+y^2}$$

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int (e^y - \sin x)\,dx$$

$$= xe^y - (-\cos x) + g(y)$$

$$F(x,y) = xe^y + \cos x + g(y)$$

So $$\frac{\partial F}{\partial y} = xe^y + 0 + g'(y)$$

$$= xe^y + g'(y)$$

Matching gives

$$x e^y + g'(y) = x e^y + \frac{1}{1+y^2}$$

$$\Rightarrow \quad g'(y) = \frac{1}{1+y^2}$$

$$g(y) = \int \frac{1}{1+y^2}\, dy = \tan^{-1} y$$

(in general $+C$)

So $\quad F(x,y) = x e^y + \cos x + \tan^{-1} y$

The solutions are given implicitly by $F(x,y) = C$

i.e. $\quad x e^y + \cos x + \tan^{-1} y = C$

Special Integrating Factors

Suppose that the equation $M\,dx + N\,dy = 0$ is not exact. Clearly our approach to exact equations would be fruitless as there is no such function F to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x,y) = xy^2$.

$$xy^2(2y-6x)\,dx + xy^2(3x-4x^2y^{-1})\,dy = 0, \quad \Longrightarrow$$

$$\underbrace{(2xy^3-6x^2y^2)}_{\text{new } M \,=\, \mu \text{ old } M}\,dx + \underbrace{(3x^2y^2-4x^3y)}_{\text{new } N \,=\, \mu \text{ old } N}\,dy = 0$$

$$\frac{\partial(\mu M)}{\partial y} = 6xy^2 - 12x^2y \qquad \frac{\partial(\mu N)}{\partial x} = 6xy^2 - 12x^2y$$

This new equation is exact.

Special Integrating Factors

The function μ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for μ of a certain *form* (usually $\mu = x^n y^m$ for some powers n and m). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on x, or there is an integrating factor $\mu = \mu(y)$ depending only on y.

Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M\,dx + N\,dy = 0$$

is NOT exact, but that

$$\mu M\,dx + \mu N\,dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on y. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

$$\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$$

Let's use the product rule in the right side.

$$\frac{\partial \mu}{\partial y} = 0$$

Special Integrating Factor $\mu = \mu(x)$

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

$$\mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x} + \frac{d\mu}{dx}N$$ Looks like a DE for μ

$$\frac{d\mu}{dx}N = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx}N = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu$$

if $N \neq 0$ $$\frac{d\mu}{dx} = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right)\mu$$

This gives a test for the existence of such a μ.

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ still has y's in it

the approach Fails!

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ depends only on x, then

μ exists and μ solves

$$\frac{d\mu}{dx} = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) \mu$$

Separate variables

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\int \frac{1}{\mu}\, d\mu = \int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\, dx$$

$$\ln \mu = \int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) dx$$

$$\mu = e^{\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) dx}$$

Special Integrating Factor

$$M\,dx + N\,dy = 0 \tag{2}$$

Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

is an special integrating factor for (2). If $(\partial N/\partial x - \partial M/\partial y)/M$ is

continuous and depends only on y, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

is an special integrating factor for (2).

Example

Solve the equation $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

$M = 2xy$, $N = y^2 - 3x^2$

$\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = -6x$ $\qquad \frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$

Is there a $\mu(x)$? $\qquad \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2x - (-6x)}{y^2 - 3x^2} = \dfrac{8x}{y^2 - 3x^2}$

Still has y in it. No $\mu(x)$ exists.

Is there a $\mu(y)$?

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x - 2x}{2xy} = \frac{-8x}{2xy} = \frac{-4}{y}$$

This depends only on y. $\mu(y)$ exists!

$$\mu(y) = e^{\int \frac{-4}{y}\,dy} = e^{-4\ln|y|} = e^{\ln y^{-4}}$$

$$\mu(y) = y^{-4}$$

We'll finish on Tuesday.