August 31 Math 3260 sec. 57 Fall 2017

Section 1.9: The Matrix for a Linear Transformation

Definition: A transformation T (a.k.a. **function** or **mapping**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Definition: A transformation T is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

A Theorem About Linear Transformations:

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0},$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for scalars c, d and vectors $\mathbf{u}.\mathbf{v}$.

And in fact

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k).$$

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]\,.$$

The matrix A is called the **standard matrix** for the linear transformation T.

One to One, Onto

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$–i.e. if the range of T is all of the codomain.

T is onto if $T(\vec{x}) = \vec{b}$ is solvable for every $\vec{b}$ in $\mathbb{R}^m$

ie. $A\vec{x} = \vec{b}$ is always consistent. where $T(\vec{x}) = A\vec{x}$

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

T is one to one if $T(\vec{x}) = T(\vec{y})$ implies $\vec{x} = \vec{y}$.

Determine if the transformation is one to one, onto, neither or both.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}. \quad = A\vec{x}$$

Is T one to one? Could $T(\vec{x}) = T(\vec{y})$ for some $\vec{x}, \vec{y}$ in $\mathbb{R}^3$.

$T(\vec{x}) = T(\vec{y}) \Rightarrow T(\vec{x}) - T(\vec{y}) = \vec{0}$

$\Rightarrow T(\vec{x} - \vec{y}) = \vec{0}$

we can see if $\vec{x} - \vec{y}$ has to be $\vec{0}$ or not.

we have the homogeneous equation

$$A\vec{x} = \vec{0}$$

Take the augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$ which is an rref

Solutions $\vec{x}$ would satisfy

$$\left.\begin{array}{lll} x_1 + 2x_3 = 0 & \Rightarrow & x_1 = -2x_3 \\ x_2 + 3x_3 = 0 & \Rightarrow & x_2 = -3x_3 \\ & & x_3 - \text{free} \end{array}\right\} \Rightarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Letting $x_3 = -1$, we get one nontrivial solution

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

$$T(\vec{x}) = \vec{0} = T(\vec{0}) \quad \text{but} \quad \vec{x} \neq \vec{0}$$

T is not one to one.

Is T onto? Is $T(\vec{x}) = \vec{b}$ solvable for all $\vec{b}$ in $\mathbb{R}^2$.

Is $A\vec{x} = \vec{b}$ always consistent?

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be any vector in $\mathbb{R}^2$.

We have augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{array}\right]$$

This implies that

$$x_1 + 2x_3 = b_1$$
$$x_2 + 3x_3 = b_2$$

For any b_1, b_2 we have a solution

$$\vec{x} = \begin{bmatrix} b_1 - 2x_3 \\ b_2 - 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

note: this looks like $\vec{p} + \vec{v}_h$

We find that $T(\vec{x}) = \vec{b}$ is solvable for any $\vec{b}$ in $\mathbb{R}^2$. Hence T is onto.

Some Theorems

Theorem : Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

This follows from

$T(\vec{x}) = T(\vec{y})$ corresponding to the

homogeneous equation

$$T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = \vec{0}$$

Some Theorems

Theorem : Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

(ii) Follows from $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$

so $A\vec{x} = \vec{0}$ has only the trivial solution

if $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly independent.

(i) Follows from the definition of $\mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

Suppose $T(x_1, x_2) = (0,0,0)$

$$(x_1, 2x_1 - x_2, 3x_2) = (0,0,0)$$

$$\Rightarrow \quad x_1 = 0, \quad 2x_1 - x_2 = 0 \quad \text{and} \quad 3x_2 = 0$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$x_1 = 0 \qquad\qquad\qquad x_2 = 0$$

This requires $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ in $\mathbb{R}^2$.

The homogeneous equation has only the trivial solution.

To determine if it's onto, lets construct the standard matrix A.

$$T(\vec{e}_1) = T(1,0) = (1, 2\cdot1-0, 3\cdot0) = (1,2,0)$$

$$T(\vec{e}_2) = T(0,1) = (0, 2\cdot0-1, 3\cdot1) = (0,-1,3)$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ consider $A\vec{x} = \vec{b}$

We have augmented matrix

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \quad -2R_1 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & -2b_1 + b_2 \\ 0 & 3 & b_3 \end{bmatrix}$$

$3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & -2b_1 + b_2 \\ 0 & 0 & -6b_1 + 3b_2 + b_3 \end{bmatrix}$$

This is only consistent if $-6b_1 + 3b_2 + b_3 = 0$

Not all vectors in $\mathbb{R}^3$ satisfy this. T is <u>not</u> Onto.

The range of T is the subset of $\mathbb{R}^3$ such that

$$\vec{b} = \begin{bmatrix} \frac{1}{2}b_2 + \frac{1}{6}b_3 \\ b_2 \\ b_3 \end{bmatrix} = b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} \frac{1}{6} \\ 0 \\ 1 \end{bmatrix}$$

This is the span of the columns of A.

Section 2.1: Matrix Operations

Recall the convenient notaton for a matrix A

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Here each column is a vector $\mathbf{a}_j$ in $\mathbb{R}^m$. We'll use the additional convenient notation to refer to A by entries

a

$$A = [a_{ij}].$$

I is a diagonal matrix

a_{ij} is the entry in **row** i and **column** j.
Main Diagonal: The main diagonal consist of the entries a_{ii}. A **diagonal matrix** is a square matrix $m = n$ for which all entries **not** on the main diagonal are zero.

Scalar Multiplication, Matrix Addition, & Equality

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate each expression or state why it fails to exist.

(a) 3B

$$3B = 3\begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 21 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

(b) A + B

$$A+B = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}$$

(c) C + A

This is undefined since A and C are of different sizes.