Section 1.9: The Matrix for a Linear Transformation

**Definition:** A transformation $T$ (a.k.a. function or mapping) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $x$ in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

**Definition:** A transformation $T$ is **linear** provided

(i) $T(u + v) = T(u) + T(v)$ for every $u, v$ in the domain of $T$, and

(ii) $T(cu) = cT(u)$ for every scalar $c$ and vector $u$ in the domain of $T$. 
A Theorem About Linear Transformations:

If \( T \) is a linear transformation, then

\[
T(0) = 0,
\]

\[
T(cu + dv) = cT(u) + dT(v)
\]

for scalars \( c, d \) and vectors \( u, v \).

And in fact

\[
T(c_1u_1 + c_2u_2 + \cdots + c_ku_k) = c_1T(u_1) + c_2T(u_2) + \cdots + c_kT(u_k).
\]
Theorem

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix $A$ such that

$$T(x) = Ax \quad \text{for every} \quad x \in \mathbb{R}^n.$$ 

Moreover, the $j^{th}$ column of the matrix $A$ is the vector $T(e_j)$, where $e_j$ is the $j^{th}$ column of the $n \times n$ identity matrix $I_n$. That is,

$$A = [T(e_1) \ T(e_2) \ \cdots \ T(e_n)].$$

The matrix $A$ is called the **standard matrix** for the linear transformation $T$. 

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One to One, Onto

**Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $b$ in $\mathbb{R}^m$ is the image of at least one $x$ in $\mathbb{R}^n$—i.e. if the range of $T$ is all of the codomain.

$$T \text{ is onto if } T(\vec{x}) = \vec{b} \text{ is solvable for all } \vec{b} \text{ in } \mathbb{R}^m$$

$$\text{i.e. if } A\vec{x} = \vec{b} \text{ is consistent for all } \vec{b} \text{ in } \mathbb{R}^m$$

where $A$ is the standard matrix for $T$.

**Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one to one** if each $b$ in $\mathbb{R}^m$ is the image of **at most one** $x$ in $\mathbb{R}^n$.

$$\text{i.e. } T \text{ is one to one if } T(\vec{x}) = T(\vec{y})$$

implies $$\vec{x} = \vec{y}$$
Determine if the transformation is one to one, onto, neither or both.

\[ T(x) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} x. \]

We already determined that \( T \) is not one to one. We found that \( T(x) = 0 \) where \( x = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \). Since it is also the case that \( T(0) = 0 \) and clearly \( x \neq 0 \), this demonstrates that \( T \) is not one to one. We ended before considering whether \( T \) is onto.

Recall that our approach was based on the observation that \( T(x) = T(y) \) would require (by virtue of linearity) \( T(x - y) = 0 \). This raised the question of whether or not it was necessary to have \( x - y = 0 \).
Is $T$ onto? We can see if $Ax = b$ is always consistent. Let $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

$Ax = b$ has augmented matrix

$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$ already in rref.

Solutions satisfy $x_1 + 2x_3 = b_1$

$x_2 + 3x_3 = b_2$

$x_3$ is free.

The system is consistent for any $b$ in $\mathbb{R}^2$ hence

$T$ is onto.
Some Theorems

Theorem: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then $T$ is one to one if and only if the homogeneous equation $T(x) = 0$ has only the trivial solution.

This follows from $T(\bar{x}) = T(\bar{y})$ corresponding to $T(\bar{x}) - T(\bar{y}) = T(\bar{x} - \bar{y}) = 0$.

If $\bar{x} - \bar{y}$ must be 0, then $\bar{x} = \bar{y}$ necessarily.
Some Theorems

**Theorem:** Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let $A$ be the standard matrix for $T$. Then

(i) $T$ is onto if and only if the columns of $A$ span $\mathbb{R}^m$, and

(ii) $T$ is one to one if and only if the columns of $A$ are linearly independent.

(ii) Follows from the definition of $A\bar{x}$ as $x_1\bar{a}_1 + x_2\bar{a}_2 + \ldots + x_n\bar{a}_n$, and the definition of linear independence.

(i) Follows from the definition of $\text{Span} \{\bar{a}_1, \ldots, \bar{a}_n\}$. If this is all of $\mathbb{R}^m$
then every $b$ in $\mathbb{R}^n$ can be written as

$$x_1 \hat{a}_1 + x_2 \hat{a}_2 + \ldots + x_n \hat{a}_n = A \hat{x}$$

for some $\hat{x}$ in $\mathbb{R}^n$. 

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that $T$ is one to one. Is $T$ onto?

To show it’s one to one consider $T(\vec{x}) = \vec{0}$.

$T(x_1, x_2) = (0, 0, 0) \Rightarrow (x_1, 2x_1 - x_2, 3x_2) = (0, 0, 0)$

$\Rightarrow x_1 = 0$ and $2x_1 - x_2 = 0$ and $3x_2 = 0$

$\Rightarrow x_1 = 0$ and $x_2 = 0$

so $x_1 = 0$ and $x_2 = 0$.

$T(\vec{x}) = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

To see if it’s onto, let’s construct the standard matrix $A$. We need $T(\vec{e}_1)$ and $T(\vec{e}_2)$.
\[ T(e_1) = T(1, 0) = (1, 2.1 - 0, 3.0) = (1, 2, 0) \]
\[ T(e_2) = T(0, 1) = (0, 2.0 - 1, 3.1) = (0, -1, 3) \]

\[
A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}
\]

Let \( \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \)

\[
A \vec{x} = \vec{b}
\]

Has Augmented matrix

\[
\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix}
\]

\[ 3R_2 + R_3 \rightarrow R_3 \]

\[
\begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & -2b_1 + b_2 \\ 0 & 0 & -6b_1 + 3b_2 + b_3 \end{bmatrix}
\]
This is only consistent if \(-6b_1 + 3b_2 + b_3 = 0\).

So, \(T\) is not onto.

We can see that the range of \(T\) is the vectors in \(\mathbb{R}^3\) such that

\[
b_1 = \frac{1}{2} b_2 + \frac{1}{6} b_3
\]

i.e.,

\[
b = \begin{bmatrix}
\frac{1}{2} b_2 + \frac{1}{6} b_3 \\
b_2 \\
b_3
\end{bmatrix} = b_2 \begin{bmatrix} \frac{1}{2} \\
1 \\
0
\end{bmatrix} + b_3 \begin{bmatrix} \frac{1}{6} \\
0 \\
1
\end{bmatrix}
\]

for \(b_2, b_3 \in \mathbb{R}\).

The range of \(T\) is

\[
\text{Span } \left\{ \begin{bmatrix} \frac{1}{2} \\
1 \\
0
\end{bmatrix}, \begin{bmatrix} \frac{1}{6} \\
0 \\
1
\end{bmatrix} \right\}
\]
Section 2.1: Matrix Operations

Recall the convenient notation for a matrix $A$

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$ 

Here each column is a vector $a_j$ in $\mathbb{R}^m$. We’ll use the additional convenient notation to refer to $A$ by entries

$$A = [a_{ij}].$$

$a_{ij}$ is the entry in row $i$ and column $j$.

**Main Diagonal:** The main diagonal consist of the entries $a_{ii}$. A **diagonal matrix** is a square matrix $m = n$ for which all entries not on the main diagonal are zero.

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The identity matrix $I$ is a diagonal matrix.
Scalar Multiplication, Matrix Addition, & Equality

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar $c$

\[ cA = [ca_{ij}] \].


Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

\[ A + B = [a_{ij} + b_{ij}] \].

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

\[ a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n. \]
Example

\[
A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}
\]

Evaluate each expression or state why it fails to exist.

(a) 3B

\[
3B = 3 \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 21 & 0 \end{bmatrix}
\]
\[ A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} \]

(b) \[ A + B = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix} \]

(c) \[ C + A \text{ undefined since } A \text{ and } C \text{ are not the same size.} \]
Theorem: Properties

The \( m \times n \) zero matrix has a zero in each entry. We’ll denote this matrix as \( O \) (or \( O_{m,n} \) if the size is not clear from the context).

Theorem: Let \( A, B, \) and \( C \) be matrices of the same size and \( r \) and \( s \) be scalars. Then

\[
\begin{align*}
(i) \quad & A + B = B + A & (iv) \quad & r(A + B) = rA + rB \\
(ii) \quad & (A + B) + C = A + (B + C) & (v) \quad & (r + s)A = rA + sA \\
(iii) \quad & A + O = A & (vi) \quad & r(sA) = (rs)A \quad \cdot (sc A)
\end{align*}
\]
Matrix Multiplication

We know that for any $m \times n$ matrix $A$, the operation "multiply vectors in $\mathbb{R}^n$ by $A$" defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to function composition. Thus if

$$S(x) = Bx,$$

and

$$T(v) = Av,$$

then

$$(T \circ S)(x) = T(S(x)) = A(Bx) = (AB)x.$$
Matrix Multiplication

Figure: Composition requires the number of rows of $B$ match the number of columns of $A$. Otherwise the product is not defined.

$$A \begin{bmatrix} B \\ \end{bmatrix}$$
Matrix Multiplication

\[ S : \mathbb{R}^p \rightarrow \mathbb{R}^n \quad \Rightarrow \quad B \sim n \times p \]
\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \Rightarrow \quad A \sim m \times n \]
\[ T \circ S : \mathbb{R}^p \rightarrow \mathbb{R}^m \quad \Rightarrow \quad AB \sim m \times p \]

\[ Bx = x_1 b_1 + x_2 b_2 + \cdots + x_p b_p \quad \Rightarrow \quad A(Bx) = x_1 Ab_1 + x_2 Ab_2 + \cdots + x_p Ab_p \quad \Rightarrow \]

\[ AB = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p] \]

The \( j^{th} \) column of \( AB \) is \( A \) times the \( j^{th} \) column of \( B \).