August 31 Math 3260 sec. 58 Fall 2017

Section 1.9: The Matrix for a Linear Transformation

Definition: A transformation T (a.k.a. **function** or **mapping**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Definition: A transformation T is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

A Theorem About Linear Transformations:

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0},$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for scalars c, d and vectors $\mathbf{u}.\mathbf{v}$.

And in fact

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1 T(\mathbf{u}_1) + c_2 T(\mathbf{u}_2) + \cdots + c_k T(\mathbf{u}_k).$$

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)].$$

The matrix A is called the **standard matrix** for the linear transformation T.

One to One, Onto

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of at least one **x** in $\mathbb{R}^n$–i.e. if the range of T is all of the codomain.

T is onto if $T(\vec{x}) = \vec{b}$ is solvable for all $\vec{b}$ in $\mathbb{R}^m$

ie. if $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^m$

where A is the standard matrix for T.

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

i.e. T is one to one if $T(\vec{x}) = T(\vec{y})$

implies $\vec{x} = \vec{y}$

Determine if the transformation is one to one, onto, neither or both.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

We already determined that T is not one to one. We found that $T(\mathbf{x}) = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Since it is also the case that $T(\mathbf{0}) = \mathbf{0}$ and clearly $\mathbf{x} \neq \mathbf{0}$, this demonstrates that T is not one to one. We ended before considering whether T is onto.

Recall that our approach was based on the observation that $T(\mathbf{x}) = T(\mathbf{y})$ would require (by virtue of linearity) $T(\mathbf{x} - \mathbf{y}) = \mathbf{0}$. This raised the question of whether or not it was necessary to have $\mathbf{x} - \mathbf{y} = \mathbf{0}$.

Is T onto? We can see if $A\vec{x}=\vec{b}$ is always consistent. Let $\vec{b}=\begin{bmatrix} b_1\\ b_2\end{bmatrix}$.

$A\vec{x}=\vec{b}$ has augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & b_1\\ 0 & 1 & 3 & b_2\end{bmatrix}$$ already in rref

Solutions satisfy
$$\begin{aligned} x_1 \quad + 2x_3 &= b_1\\ x_2 + 3x_3 &= b_2\\ x_3 &- \text{free}\end{aligned}$$

The system is consistent for any $\vec{b}$ in $\mathbb{R}^2$ hence

T is onto.

Some Theorems

Theorem : Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

This follows from $T(\vec{x}) = T(\vec{y})$ corresponding

to $T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = \vec{0}$

If $\vec{x} - \vec{y}$ must be $\vec{0}$, then $\vec{x} = \vec{y}$

necessarily.

Some Theorems

Theorem : Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

(ii) Follows from the definition of $A\vec{x}$ as

$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$ and the definition

of linear independence.

(i) Follows from the definition of

$\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$. If this is all of $\mathbb{R}^m$

then every $\vec{b}$ in $\mathbb{R}^n$ can be written as

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = A\vec{x}$$

for some $\vec{x}$ in $\mathbb{R}^n$.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

To show it's one to one consider $T(\vec{x}) = \vec{0}$.

$T(x_1, x_2) = (0,0,0) \Rightarrow (x_1, 2x_1 - x_2, 3x_2) = (0,0,0)$

$\Rightarrow x_1 = 0$ and $2x_1 - x_2 = 0$ and $3x_2 = 0$

so $x_1 = 0$ and $x_2 = 0$

$T(\vec{x}) = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

To see if it's onto, lets construct the standard matrix A. We need $T(\vec{e}_1)$ and $T(\vec{e}_2)$

$T(\vec{e}_1) = T(1,0) = (1, 2\cdot 1 - 0, 3\cdot 0) = (1,2,0)$

$T(\vec{e}_2) = T(0,1) = (0, 2\cdot 0 - 1, 3\cdot 1) = (0,-1,3)$

$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$. Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$A\vec{x} = \vec{b}$ has Augmented matrix

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \quad -2R_1 + R_2 \to R_2 \quad \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & -2b_1 + b_2 \\ 0 & 3 & b_3 \end{bmatrix}$$

$$3R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & -2b_1 + b_2 \\ 0 & 0 & -6b_1 + 3b_2 + b_3 \end{bmatrix}$$

This is only consistent if $-6b_1+3b_2+b_3=0$.

So T is <u>not</u> onto.

We can see that the range of T is the vectors in $\mathbb{R}^3$ such that $b_1 = \frac{1}{2}b_2 + \frac{1}{6}b_3$

i.e. $\vec{b} = \begin{bmatrix} \frac{1}{2}b_2+\frac{1}{6}b_3 \\ b_2 \\ b_3 \end{bmatrix} = b_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} \frac{1}{6} \\ 0 \\ 1 \end{bmatrix}$ for b_2, b_3 in $\mathbb{R}$

The range of T is

$$\mathrm{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/6 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Section 2.1: Matrix Operations

Recall the convenient notaton for a matrix A

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Here each column is a vector $\mathbf{a}_j$ in $\mathbb{R}^m$. We'll use the additional convenient notation to refer to A by entries

$$A = [a_{ij}].$$

I is the identity a diagonal matrix

a_{ij} is the entry in **row** i and **column** j.
Main Diagonal: The main diagonal consist of the entries a_{ii}. A **diagonal matrix** is a square matrix $m = n$ for which all entries **not** on the main diagonal are zero.

Scalar Multiplication, Matrix Addition, & Equality

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate each expression or state why it fails to exist.

(a) 3B

$$3B = 3\begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 21 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

(b) A + B $= \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}$

(c) C + A   undefined since A and C are not the same size.

Theorem: Properties

The $m \times n$ **zero matrix** has a zero in each entry. We'll denote this matrix as O (or $O_{m,n}$ if the size is not clear from the context).

Theorem: Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$	(iv) $r(A + B) = rA + rB$
(ii) $(A + B) + C = A + (B + C)$	(v) $(r + s)A = rA + sA$
(iii) $A + O = A$	(vi) $r(sA) = (rs)A$ $= (sr)A$

Matrix Multiplication

We know that for any $m \times n$ matrix A, the operation **"multiply vectors in $\mathbb{R}^n$ by A"** defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to **function composition.** Thus if

$$S(\mathbf{x}) = B\mathbf{x}, \quad \text{and} \quad T(\mathbf{v}) = A\mathbf{v},$$

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix Multiplication

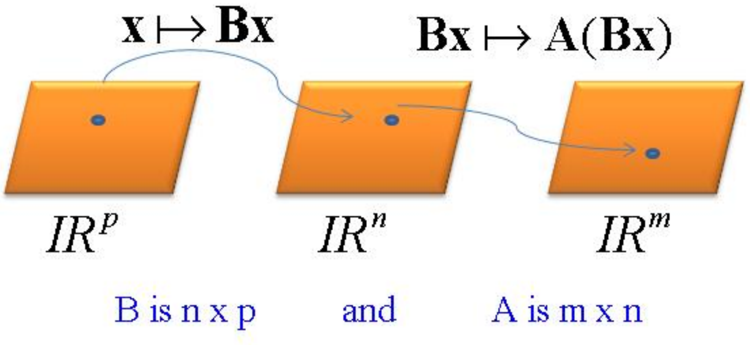

Figure: Composition requires the number of rows of B match the number of columns of A. Otherwise the product is **not defined**.

$$\underset{m\times n}{A}\ \underset{n\times p}{B}$$

Matrix Multiplication

$$S : \mathbb{R}^p \longrightarrow \mathbb{R}^n \quad \Longrightarrow \quad B \sim n \times p$$

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad A \sim m \times n$$

$$T \circ S : \mathbb{R}^p \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad AB \sim m \times p$$

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p \Longrightarrow$$

$$A(B\mathbf{x}) = x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p \Longrightarrow$$

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

The j^{th} column of AB is A times the j^{th} column of B.