Calculus Basics

Assume that f and g are integrable functions and that k is a nonzero constant.

$$
\begin{aligned}
\int 1\,dx &= x + C \\
\int x^n\,dx &= \frac{x^{n+1}}{n+1} + C \quad n \neq -1 \\
\int \sin(kx)\,dx &= -\frac{1}{k}\cos(kx) + C \\
\int \cos(kx)\,dx &= \frac{1}{k}\sin(kx) + C \\
\int \sec^2(kx)\,dx &= \frac{1}{k}\tan(kx) + C \\
\int \csc^2(kx)\,dx &= -\frac{1}{k}\cot(kx) + C \\
\int \sec(kx)\tan(kx)\,dx &= \frac{1}{k}\sec(kx) + C \\
\int \csc(kx)\cot(kx)\,dx &= -\frac{1}{k}\csc(kx) + C \\
\int \tan x\,dx &= \ln|\sec x| + C \\
\int \cot x\,dx &= \ln|\sin x| + C \\
\int \sec x\,dx &= \ln|\sec x + \tan x| + C \\
\int \csc x\,dx &= -\ln|\csc x + \cot x| + C \\
\int (f(x) \pm g(x))\,dx &= \int f(x)\,dx \pm \int g(x)\,dx \\
\int kf(x)\,dx &= k\int f(x)\,dx \\
\int \frac{1}{u}\,du &= \ln|u| + C \\
\int e^u\,du &= e^u + C
\end{aligned}
$$

$$\begin{aligned}
\frac{d}{dx}\sin^{-1}x &= \frac{1}{\sqrt{1-x^2}} = -\frac{d}{dx}\cos^{-1}x \\
\frac{d}{dx}\sec^{-1}x &= \frac{1}{|x|\sqrt{x^2-1}} = -\frac{d}{dx}\csc^{-1}x \\
\frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2} = -\frac{d}{dx}\cot^{-1}x \\
\int \frac{1}{\sqrt{a^2-u^2}}\,du &= \sin^{-1}\left(\frac{u}{a}\right) + C \quad a^2-u^2>0 \\
\int \frac{1}{a^2+u^2}\,du &= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \\
\int \frac{1}{u\sqrt{u^2-a^2}}\,du &= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \quad u^2-a^2>0 \\
\int u\,dv &= uv - \int v\,du \\
\sin mx \sin nx &= \frac{1}{2}(\cos(m-n)x - \cos(m+n)x) \\
\cos mx \cos nx &= \frac{1}{2}(\cos(m-n)x + \cos(m+n)x) \\
\sin mx \cos nx &= \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)
\end{aligned}$$

Some Sequence and Series Stuff

The sequence $\{a_n\}$ is said to converge to a if $\lim_{n\to\infty} a_n = a$. If the limit is infinite or otherwise does not exist, the sequence is divergent.

The infinte series $\sum a_n$ is convergent with sum s if the sequence of partial sums

$$s_n = \sum_{k=1}^{n} a_n$$

converges to s. If the limit of the partial sums doesn't exist, the series $\sum a_n$ is divergent.

Some special series:

(1) (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$ and converges if $p > 1$

(2) (geometric) $\sum_{n=0}^{\infty} ar^n$ diverges if $|r| \geq 1$. If $|r| < 1$ it converges with sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

(3) (telescoping) $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ diverges if $\lim_{n\to\infty} a_n$ doesn't exist. If this limit exists, the sum converges to $a_1 - \lim_{n\to\infty} a_n$.

nth term test (a.k.a. Divergence test): If $\lim_{n\to\infty} a_n \neq 0$, the series $\sum a_n$ is divergent.

Integral test: If f is a positive, decreasing, integrable function for $x \geq N$, and if $f(n) = a_n$ for integers $n \geq N$ then the integral and the series

$$\int_N^{\infty} f(x)\,dx \quad \text{and} \quad \sum_{n=N}^{\infty} a_n$$

both converge or both diverge.

Direct comparison test: We consider two series of nonegative terms $\sum a_n$ and $\sum b_n$. Suppose

$$0 \leq a_n \leq b_n \quad \text{for all } n \geq N.$$

Then

(1) If $\sum b_n$ converges, then $\sum a_n$ converges, and

(2) if $\sum a_n$ diverges, then $\sum b_n$ diverges.

Limit comparison test Let $\sum a_n$ and $\sum b_n$ be series of positive terms. If

(1) $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$, then both series converge or both diverge;

(2) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ converges, then $\sum a_n$ converges; and

(3) $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ratio test: Consider the series (of nonzero terms) $\sum a_n$ Let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(1) If $L < 1$, the series converges,

(2) if $L > 1$, the series diverges, and

(3) if $L = 1$, the test gives no information.

Root test: Consider the series $\sum a_n$ Let

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

(1) If $L < 1$, the series converges,

(2) if $L > 1$, the series diverges, and

(3) if $L = 1$, the test gives no information.

Alternating series test: Consider the series $\sum(-1)^n u_n$ or $\sum(-1)^{n+1} u_n$, where the terms u_n are all nonnegative. If

(1) $u_n \geq u_{n+1}$ for all $n \geq n_0$ for some index n_0, and

(2) $\lim_{n\to\infty} u_n = 0$

then the series converges.

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ is said to *converge absolutely*. If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ is said to *converge conditionally*.

If f is a function that is defined on an interval I with derivatives of all orders on I, and if a is an interior point of I, then the Taylor series generated by f centered at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $a = 0$, we call the series a Maclaurin series.

A Taylor series is a special example of a power series. A power series is one of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n.$$

If this series converges whenever $|x - a| < R$ (where R is the largest such value or infinite if the series converges for all $-\infty < x < \infty$), then we say that the *radius of convergence* is R. The interval of convergence (for finite R) is one of $a - R < x < a + R$, $a - R \leq x < a + R$, $a - R < x \leq a + R$, or $a - R \leq x \leq a + R$.

Binomial series For $-1 < x < 1$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots = \sum_{n=0}^{\infty} \binom{m}{n} x^n.$$

Here,

$$\binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}.$$

Polar Graph Integration The area of the region between the origin and the function $r = f(\theta)$ for θ between α and β is

$$\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left(f(\theta)\right)^2 \, d\theta.$$