Calculus Basics

Assume that $f$ and $g$ are integrable functions and that $k$ is a nonzero constant.

\[
\begin{align*}
\int 1 \, dx & = x + C \\
\int x^n \, dx & = \frac{x^{n+1}}{n+1} + C \quad n \neq -1 \\
\int \sin(kx) \, dx & = -\frac{1}{k} \cos(kx) + C \\
\int \cos(kx) \, dx & = \frac{1}{k} \sin(kx) + C \\
\int \sec^2(kx) \, dx & = \frac{1}{k} \tan(kx) + C \\
\int \csc^2(kx) \, dx & = -\frac{1}{k} \cot(kx) + C \\
\int \sec(kx) \tan(kx) \, dx & = \frac{1}{k} \sec(kx) + C \\
\int \csc(kx) \cot(kx) \, dx & = -\frac{1}{k} \csc(kx) + C \\
\int \tan x \, dx & = \ln |\sec x| + C \\
\int \cot x \, dx & = \ln |\sin x| + C \\
\int \sec x \, dx & = \ln |\sec x + \tan x| + C \\
\int \csc x \, dx & = -\ln |\csc x + \cot x| + C \\
\int (f(x) \pm g(x)) \, dx & = \int f(x) \, dx \pm \int g(x) \, dx \\
\int kf(x) \, dx & = k \int f(x) \, dx \\
\int \frac{1}{u} \, du & = \ln |u| + C \\
\int e^u \, du & = e^u + C
\end{align*}
\]
\[ \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} = -\frac{d}{dx} \cos^{-1} x \]
\[ \frac{d}{dx} \sec^{-1} x = \frac{1}{|x| \sqrt{x^2 - 1}} = -\frac{d}{dx} \csc^{-1} x \]
\[ \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} = -\frac{d}{dx} \cot^{-1} x \]
\[ \int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1} \left( \frac{u}{a} \right) + C \quad a^2 - u^2 > 0 \]
\[ \int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1} \left( \frac{u}{a} \right) + C \]
\[ \int \frac{1}{u \sqrt{u^2 - a^2}} \, du = \frac{1}{a} \sec^{-1} \left( \frac{|u|}{a} \right) + C \quad u^2 - a^2 > 0 \]
\[ \int u \, dv = uv - \int v \, du \]
\[ \sin mx \sin nx = \frac{1}{2} (\cos(mx - nx) - \cos(mx + nx)) \]
\[ \cos mx \cos nx = \frac{1}{2} (\cos(mx - nx) + \cos(mx + nx)) \]
\[ \sin mx \cos nx = \frac{1}{2} (\sin(mx - nx) + \sin(mx + nx)) \]

Some Sequence and Series Stuff

The sequence \( \{a_n\} \) is said to converge to \( a \) if \( \lim_{n \to \infty} a_n = a \). If the limit is infinite or otherwise does not exist, the sequence is divergent.

The infinite series \( \sum a_n \) is convergent with sum \( s \) if the sequence of partial sums
\[ s_n = \sum_{k=1}^{n} a_n \]
converges to \( s \). If the limit of the partial sums doesn’t exist, the series \( \sum a_n \) is divergent.

Some special series:

1. (p-series) \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges if \( p \leq 1 \) and converges if \( p > 1 \)

2. (geometric) \( \sum_{n=0}^{\infty} ar^n \) diverges if \( |r| \geq 1 \). If \( |r| < 1 \) it converges with sum
\[ \sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r} \]
(3) (telescoping) \( \sum_{n=1}^{\infty} (a_n - a_{n+1}) \) diverges if \( \lim_{n \to \infty} a_n \) doesn’t exist. If this limit exists, the sum converges to \( a_1 - \lim_{n \to \infty} a_n \).

\textbf{nth term test (a.k.a. Divergence test):} If \( \lim_{n \to \infty} a_n \neq 0 \), the series \( \sum a_n \) is divergent.

\textbf{Integral test:} If \( f \) is a positive, decreasing, integrable function for \( x \geq N \), and if \( f(n) = a_n \) for integers \( n \geq N \) then the integral and the series

\[
\int_{N}^{\infty} f(x) \, dx \quad \text{and} \quad \sum_{n=N}^{\infty} a_n
\]

both converge or both diverge.

\textbf{Direct comparison test:} We consider two series of nonegative terms \( \sum a_n \) and \( \sum b_n \). Suppose

\[
0 \leq a_n \leq b_n \quad \text{for all } n \geq N.
\]

Then

(1) If \( \sum b_n \) converges, then \( \sum a_n \) converges, and

(2) if \( \sum a_n \) diverges, then \( \sum b_n \) diverges.

\textbf{Limit comparison test} Let \( \sum a_n \) and \( \sum b_n \) be series of positive terms. If

(1) \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \) where \( 0 < L < \infty \), then both series converge or both diverge;

(2) \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \), and \( \sum b_n \) converges, then \( \sum a_n \) converges; and

(3) \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \), and \( \sum b_n \) diverges, then \( \sum a_n \) diverges.
**Ratio test:** Consider the series (of nonzero terms) \( \sum a_n \). Let

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

(1) If \( L < 1 \), the series converges,

(2) if \( L > 1 \), the series diverges, and

(3) if \( L = 1 \), the test gives no information.

**Root test:** Consider the series \( \sum a_n \). Let

\[
L = \lim_{n \to \infty} \sqrt[n]{|a_n|}
\]

(1) If \( L < 1 \), the series converges,

(2) if \( L > 1 \), the series diverges, and

(3) if \( L = 1 \), the test gives no information.

**Alternating series test:** Consider the series \( \sum (-1)^n u_n \) or \( \sum (-1)^{n+1} u_n \), where the terms \( u_n \) are all nonnegative. If

(1) \( u_n \geq u_{n+1} \) for all \( n \geq n_0 \) for some index \( n_0 \), and

(2) \( \lim_{n \to \infty} u_n = 0 \)

then the series converges.

If the series \( \sum |a_n| \) converges, then the series \( \sum a_n \) is said to converge **absolutely**. If \( \sum a_n \) converges, but \( \sum |a_n| \) diverges, then \( \sum a_n \) is said to converge **conditionally**.
If \( f \) is a function that is defined on an interval \( I \) with derivatives of all orders on \( I \), and if \( a \) is an interior point of \( I \), then the Taylor series generated by \( f \) centered at \( x = a \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.
\]

If \( a = 0 \), we call the series a Maclaurin series.

A Taylor series is a special example of a power series. A power series is one of the form

\[
\sum_{n=0}^{\infty} c_n (x-a)^n.
\]

If this series converges whenever \(|x-a| < R\) (where \( R \) is the largest such value or infinite if the series converges for all \(-\infty < x < \infty\)), then we say that the **radius of convergence** is \( R \).

The interval of convergence (for finite \( R \)) is one of \( a-R < x < a+R \), \( a-R \leq x < a+R \), \( a-R < x \leq a+R \), or \( a-R \leq x \leq a+R \).

**Binomial series** For \(-1 < x < 1\)

\[
(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots = \sum_{n=0}^{\infty} \binom{m}{n} x^n.
\]

Here,

\[
\binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}.
\]

**Polar Graph Integration** The area of the region between the origin and the function \( r = f(\theta) \) for \( \theta \) between \( \alpha \) and \( \beta \) is

\[
\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta.
\]