Section 8.1 *Sequences*

APEX CALCULUS II

Version 2.0

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8.1 Sequences

We commonly refer to a set of events that occur one after the other as a *sequence* of events. In mathematics, we use the word *sequence* to refer to an ordered set of numbers, i.e., a set of numbers that "occur one after the other."

For instance, the numbers 2, 4, 6, 8, ..., form a sequence. The order is important; the first number is 2, the second is 4, etc. It seems natural to seek a formula that describes a given sequence, and often this can be done. For instance, the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, \ldots$ To find the 10^{th} term in the sequence, we would compute $a(10)$. This leads us to the following, formal definition of a sequence.

Notation: We use $\mathbb{N}$ to describe the set of natural numbers, that is, the integers 1, 2, 3, ...

Factorial: The expression 3! refers to the number $3 \cdot 2 \cdot 1 = 6$.

In general, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, where n is a natural number.

We define $0! = 1$. While this does not immediately make sense, it makes many mathematical formulas work properly.

Definition 27 Sequence

A **sequence** is a function $a(n)$ whose domain is $\mathbb{N}$. The **range** of a sequence is the set of all distinct values of $a(n)$.

The **terms** of a sequence are the values $a(1)$, $a(2)$, ..., which are usually denoted with subscripts as a_1, a_2, ...

A sequence $a(n)$ is often denoted as $\{a_n\}$.

Example 228 Listing terms of a sequence

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3^n}{n!}\right\}$ 2. $\{a_n\} = \{4 + (-1)^n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^{n(n+1)/2}}{n^2}\right\}$

Solution

1. $a_1 = \dfrac{3^1}{1!} = 3;$ $a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2};$ $a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2};$ $a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$

   We can plot the terms of a sequence with a scatter plot. The "x"-axis is used for the values of n, and the values of the terms are plotted on the y-axis. To visualize this sequence, see Figure 8.1.

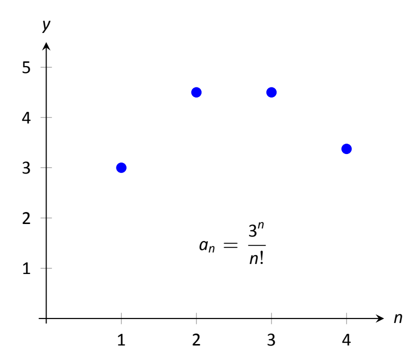
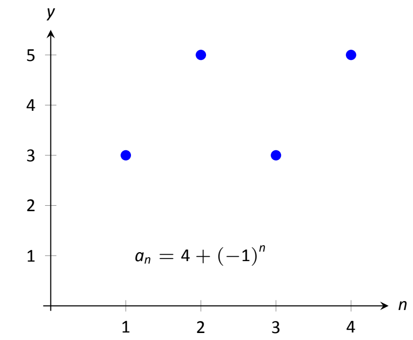
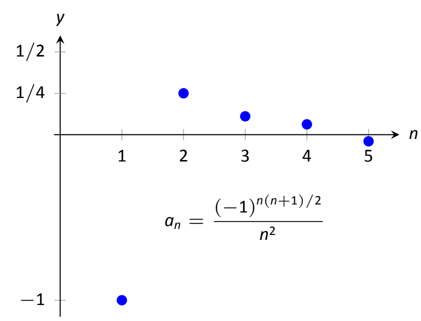


Figure 8.1: Plotting a sequence from Example 228.



(a)



(b)

Figure 8.2: Plotting sequences in Example 228.

2. $a_1 = 4 + (-1)^1 = 3$; $\quad a_2 = 4 + (-1)^2 = 5$;
   $a_3 = 4+(-1)^3 = 3$; $\quad a_4 = 4+(-1)^4 = 5$. Note that the range of this sequence is finite, consisting of only the values 3 and 5. This sequence is plotted in Figure 8.2 (a).

3. $a_1 = \dfrac{(-1)^{1(2)/2}}{1^2} = -1$; $\quad a_2 = \dfrac{(-1)^{2(3)/2}}{2^2} = -\dfrac{1}{4}$
   $a_3 = \dfrac{(-1)^{3(4)/2}}{3^2} = \dfrac{1}{9}$ $\quad a_4 = \dfrac{(-1)^{4(5)/2}}{4^2} = \dfrac{1}{16}$;
   $a_5 = \dfrac{(-1)^{5(6)/2}}{5^2} = -\dfrac{1}{25}$.
   We gave one extra term to begin to show the pattern of signs is "$-, -, +, +, -, -, \ldots$, due to the fact that the exponent of -1 is a special quadratic. This sequence is plotted in Figure 8.2 (b).

Example 229 Determining a formula for a sequence

Find the n^{th} term of the following sequences, i.e., find a function that describes each of the given sequences.

1. $2, 5, 8, 11, 14, \ldots$
2. $2, -5, 10, -17, 26, -37, \ldots$
3. $1, 1, 2, 6, 24, 120, 720, \ldots$
4. $\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{15}{8}, \dfrac{5}{4}, \dfrac{25}{32}, \ldots$

Solution We should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with 2, then 5, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is 3 more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of b. As we want $a_1 = 2$, we set $b = -1$. Thus $a_n = 3n - 1$.

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-1)^n$ or $(-1)^{n+1}$. Using $(-1)^n$ multiplies the odd terms by (-1); using $(-1)^{n+1}$ multiplies the even terms by (-1). As this sequence has negative even terms, we will multiply by $(-1)^{n+1}$.

Notes:

After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers 2, 5, 10, 17, etc., have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $2 = 1^1 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, etc. Thus our formula is $a_n = (-1)^{n+1}(n^2 + 1)$.

3. One who is familiar with the factorial function will readily recognize these numbers. They are 0!, 1!, 2!, 3!, etc. Since our sequences start with $n = 1$, we cannot write $a_n = n!$, for this misses the 0! term. Instead, we shift by 1, and write $a_n = (n-1)!$.

4. This one may appear difficult, especially as the first two terms are the same, but a little "sleuthing" will help. Notice how the terms in the numerator are always multiples of 5, and the terms in the denominator are always powers of 2. Does something as simple as $a_n = \frac{5n}{2^n}$ work?

   When $n = 1$, we see that we indeed get $5/2$ as desired. When $n = 2$, we get $10/4 = 5/2$. Further checking shows that this formula indeed matches the other terms of the sequence.

A common mathematical endeavor is to create a new mathematical object (for instance, a sequence) and then apply previously known mathematics to the new object. We do so here. The fundamental concept of calculus is the limit, so we will investigate what it means to find the limit of a sequence.

Definition 28 Limit of a Sequence, Convergent, Divergent

Let $\{a_n\}$ be a sequence and let L be a real number. Given any $\varepsilon > 0$, if an m can be found such that $|a_n - L| < \varepsilon$ for all $n > m$, then we say the **limit of $\{a_n\}$, as n approaches infinity, is** L, denoted

$$\lim_{n\to\infty} a_n = L.$$

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges**; otherwise, the sequence **diverges**.

This definition states, informally, that if the limit of a sequence is L, then if you go far enough out along the sequence, all subsequent terms will be *really close* to L. Of course, the terms "far enough" and "really close" are subjective terms, but hopefully the intent is clear.

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This definition is reminiscent of the ε–δ proofs of Chapter 1. In that chapter we developed other tools to evaluate limits apart from the formal definition; we do so here as well.

Theorem 55 Limit of a Sequence

Let $\{a_n\}$ be a sequence and let $f(x)$ be a function where $f(n) = a_n$ for all n in $\mathbb{N}$.

1. If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.
2. If $\lim\limits_{x\to\infty} f(x)$ does not exist, then $\{a_n\}$ diverges.

When we considered limits before, the domain of the function was an interval of real numbers. Now, as we consider limits, the domain is restricted to $\mathbb{N}$, the natural numbers. Theorem 55 states that this restriction of the domain does not affect the outcome of the limit and whatever tools we developed in Chapter 1 to evaluate limits can be applied here as well.

Example 230 Determining convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3n^2-2n+1}{n^2-1000}\right\}$ 2. $\{a_n\} = \{\cos n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$

Solution

1. Using Theorem 11, we can state that $\lim\limits_{x\to\infty}\dfrac{3x^2-2x+1}{x^2-1000} = 3$. (We could have also directly applied l'Hôpital's Rule.) Thus the sequence $\{a_n\}$ converges, and its limit is 3. A scatter plot of every 5 values of a_n is given in Figure 8.3 (a). The values of a_n vary widely near $n = 30$, ranging from about -73 to 125, but as n grows, the values approach 3.

2. The limit $\lim\limits_{x\to\infty}\cos x$ does not exist, as the function oscillates (and takes on every value in $[-1, 1]$ infinitely many times). Thus we conclude that the sequence $\{\cos n\}$ diverges. (And in this particular case, since the domain is restricted to $\mathbb{N}$, no value of $\cos n$ is repeated!) This sequence is plotted in Figure 8.3 (b); because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave.

3. We cannot actually apply Theorem 55 here, as the function $f(x) = (-1)^x/x$

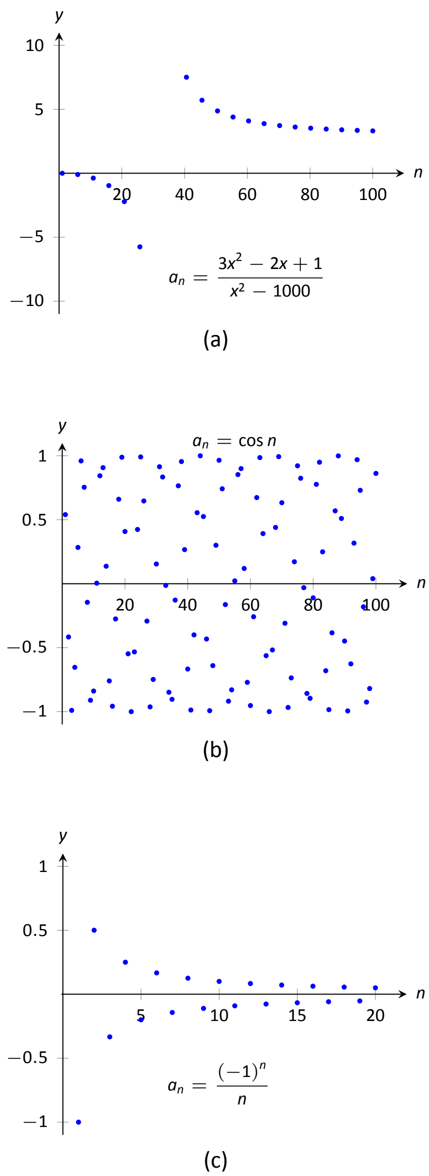

Figure 8.3: Scatter plots of the sequences in Example 230.

Notes:

is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.) So for now we say that we cannot determine the limit. (But we will be able to very soon.) By looking at the plot in Figure 8.3 (c), we would like to conclude that the sequence converges to 0. That is true, but at this point we are unable to decisively say so.

It seems very clear that a sequence such as $\left\{\dfrac{(-1)^n}{n}\right\}$ converges to 0 but we lack the formal tool to prove it. The following theorem gives us that tool.

Theorem 56 Absolute Value Theorem

Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

Example 231 Determining the convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ 2. $\{a_n\} = \left\{\dfrac{(-1)^n(n+1)}{n}\right\}$

Solution

1. This appeared in Example 230. We want to apply Theorem 56, so consider the limit of $\{|a_n|\}$:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0.\end{aligned}$$

Since this limit is 0, we can apply Theorem 56 and state that $\lim\limits_{n\to\infty} a_n = 0$.

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by -1), we cannot simply look at the limit $\lim\limits_{x\to\infty} \dfrac{(-1)^x(x+1)}{x}$. We can try to apply the techniques of Theorem 56:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n(n+1)}{n}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= 1.\end{aligned}$$

Notes:

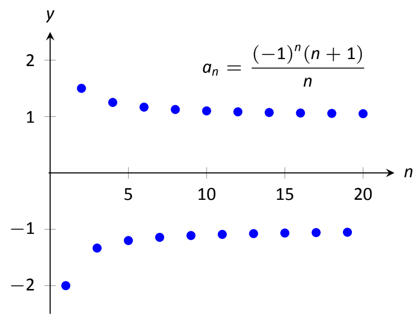

Figure 8.4: A plot of a sequence in Example 231, part 2.

We have concluded that when we ignore the alternating sign, the sequence approaches 1. This means we cannot apply Theorem 56; it states the the limit must be 0 in order to conclude anything.

In fact, since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is 1, we know that as n approaches infinity, the terms will alternate between values close to 1 and -1, meaning the sequence diverges. A plot of this sequence is given in Figure 8.4.

We continue our study of the limits of sequences by considering some of the properties of these limits.

Theorem 57 Properties of the Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = K$, and let c be a real number.

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = L \cdot K$
3. $\lim\limits_{n\to\infty} (a_n/b_n) = L/K$, $K \neq 0$
4. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot L$

Example 232 Applying properties of limits of sequences

Let the following sequences, and their limits, be given:

- $\{a_n\} = \left\{\dfrac{n+1}{n^2}\right\}$, and $\lim\limits_{n\to\infty} a_n = 0$;
- $\{b_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$, and $\lim\limits_{n\to\infty} b_n = e$; and
- $\{c_n\} = \{n \cdot \sin(5/n)\}$, and $\lim\limits_{n\to\infty} c_n = 5$.

Evaluate the following limits.

1. $\lim\limits_{n\to\infty} (a_n + b_n)$
2. $\lim\limits_{n\to\infty} (b_n \cdot c_n)$
3. $\lim\limits_{n\to\infty} (1000 \cdot a_n)$

Solution We will use Theorem 57 to answer each of these.

1. Since $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = e$, we conclude that $\lim\limits_{n\to\infty} (a_n + b_n) = 0 + e = e$. So even though we are adding something to each term of the sequence b_n, we are adding something so small that the final limit is the same as before.

Notes:

2. Since $\lim_{n\to\infty} b_n = e$ and $\lim_{n\to\infty} c_n = 5$, we conclude that $\lim_{n\to\infty}(b_n \cdot c_n) = e \cdot 5 = 5e$.

3. Since $\lim_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} 1000a_n = 1000 \cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to 0.)

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Definition 29 Bounded and Unbounded Sequences

A sequence $\{a_n\}$ is said to be **bounded** if there exists real numbers m and M such that $m < a_n < M$ for all n in $\mathbb{N}$.

A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.

A sequence $\{a_n\}$ is said to be **bounded above** if there exists an M such that $a_n < M$ for all n in $\mathbb{N}$; it is **bounded below** if there exists an m such that $m < a_n$ for all n in $\mathbb{N}$.

It follows from this definition that an unbounded sequence may be bounded above or bounded below; a sequence that is both bounded above and below is simply a bounded sequence.

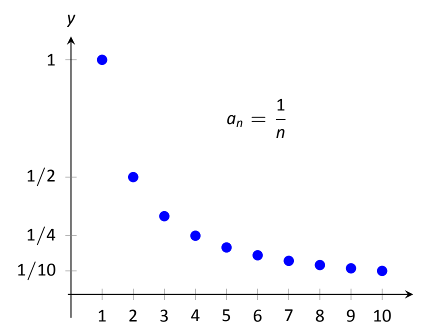

Figure 8.5: A plot of $\{a_n\} = \{1/n\}$ from Example 233.

Example 233 Determining boundedness of sequences

Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\frac{1}{n}\right\}$
2. $\{a_n\} = \{2^n\}$

Solution

1. The terms of this sequence are always positive but are decreasing, so we have $0 < a_n < 2$ for all n. Thus this sequence is bounded. Figure 8.5 illustrates this.

2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. Figure 8.6 illustrates this.

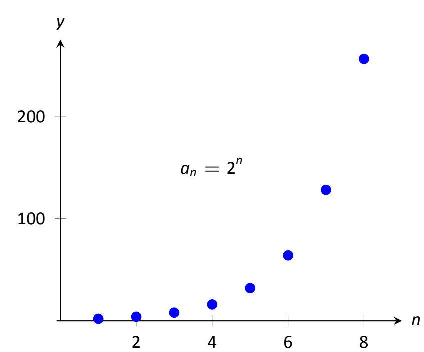

Figure 8.6: A plot of $\{a_n\} = \{2^n\}$ from Example 233.

Notes:

The previous example produces some interesting concepts. First, we can recognize that the sequence $\{1/n\}$ converges to 0. This says, informally, that "most" of the terms of the sequence are "really close" to 0. This implies that the sequence is bounded, using the following logic. First, "most" terms are near 0, so we could find some sort of bound on these terms (using Definition 28, the bound is ε). That leaves a "few" terms that are not near 0 (i.e., a *finite* number of terms). A finite list of numbers is always bounded.

This logic implies that if a sequence converges, it must be bounded. This is indeed true, as stated by the following theorem.

Note: Keep in mind what Theorem 58 does *not* say. It does not say that bounded sequences must converge, nor does it say that if a sequence does not converge, it is not bounded.

Theorem 58 Convergent Sequences are Bounded

Let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

In Example 232 we saw the sequence $\{b_n\} = \left\{(1+1/n)^n\right\}$, where it was stated that $\lim_{n\to\infty} b_n = e$. (Note that this is simply restating part of Theorem 5.) Even though it may be difficult to intuitively grasp the behavior of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of Example 233 again involves the sequence $\{1/n\}$. We stated, without proof, that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n+1$. Taking reciprocals flips the inequality: $1/n > 1/(n+1)$. This is the same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Note: It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all n; i.e, we remove the possibility that subsequent terms are equal.
A similar statement holds for *strictly decreasing.*

Definition 30 Monotonic Sequences

1. A sequence $\{a_n\}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all n, i.e.,
$$a_1 \leq a_2 \leq a_3 \leq \cdots a_n \leq a_{n+1} \cdots$$
2. A sequence $\{a_n\}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all n, i.e.,
$$a_1 \geq a_2 \geq a_3 \geq \cdots a_n \geq a_{n+1} \cdots$$
3. A sequence is **monotonic** if it is monotonically increasing or monotonically decreasing.

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Example 234 **Determining monotonicity**
Determine the monotonicity of the following sequences.

1. $\{a_n\} = \left\{\dfrac{n+1}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{n^2+1}{n+1}\right\}$
3. $\{a_n\} = \left\{\dfrac{n^2-9}{n^2-10n+26}\right\}$
4. $\{a_n\} = \left\{\dfrac{n^2}{n!}\right\}$

Solution In each of the following, we will examine $a_{n+1}-a_n$. If $a_{n+1}-a_n > 0$, we conclude that $a_n < a_{n+1}$ and hence the sequence is increasing. If $a_{n+1}-a_n < 0$, we conclude that $a_n > a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.

We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

1.
$$\begin{aligned} a_{n+1}-a_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\ &= \frac{(n+2)(n)-(n+1)^2}{(n+1)n} \\ &= \frac{-1}{n(n+1)} \\ &< 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1}-a_n < 0$ for all n, we conclude that the sequence is decreasing.

2.
$$\begin{aligned} a_{n+1}-a_n &= \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1} \\ &= \frac{\big((n+1)^2+1\big)(n+1)-(n^2+1)(n+2)}{(n+1)(n+2)} \\ &= \frac{n^2+4n+1}{(n+1)(n+2)} \\ &> 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1}-a_n > 0$ for all n, we conclude the sequence is increasing.

3. We can clearly see in Figure 8.7 (c), where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

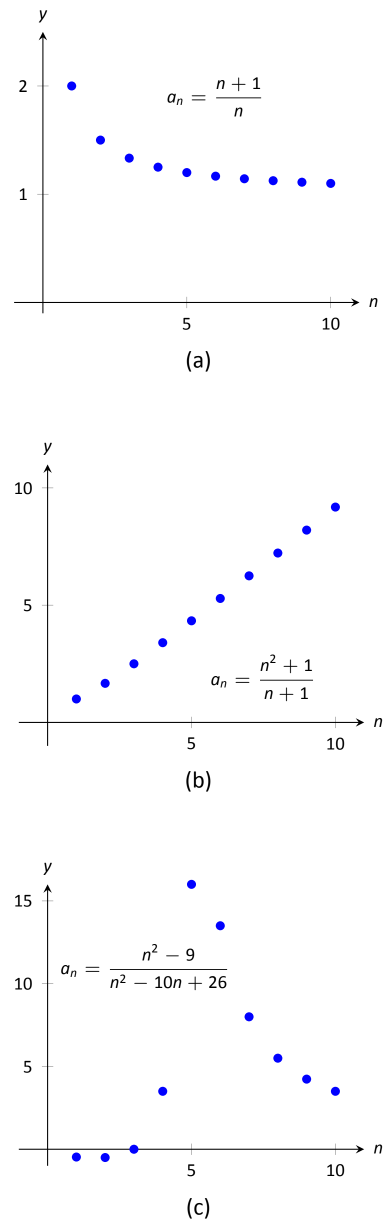

Figure 8.7: Plots of sequences in Example 234.

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$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2 - 9}{(n+1)^2 - 10(n+1) + 26} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{n^2 + 2n - 8}{n^2 - 8n + 17} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{(n^2 + 2n - 8)(n^2 - 10n + 26) - (n^2 - 9)(n^2 - 8n + 17)}{(n^2 - 8n + 17)(n^2 - 10n + 26)} \\ &= \frac{-10n^2 + 60n - 55}{(n^2 - 8n + 17)(n^2 - 10n + 26)}. \end{aligned}$$

We want to know when this is greater than, or less than, 0, therefore we are only concerned with the numerator. Using the quadratic formula, we can determine that $-10n^2 + 60n - 55 = 0$ when $n \approx 1.13, 4.87$. So for $n < 1.13$, the sequence is decreasing. Since we are only dealing with the natural numbers, this means that $a_1 > a_2$.

Between 1.13 and 4.87, i.e., for $n = 2$, 3 and 4, we have that $a_{n+1} > a_n$ and the sequence is increasing. (That is, when $n = 2$, 3 and 4, the numerator $-10n^2 + 60n + 55$ from the fraction above is > 0.)

When $n > 4.87$, i.e, for $n \geq 5$, we have that $-10n^2 + 60n + 55 < 0$, hence $a_{n+1} - a_n < 0$, so the sequence is decreasing.

In short, the sequence is simply not monotonic. However, it is useful to note that for $n \geq 5$, the sequence is monotonically decreasing.

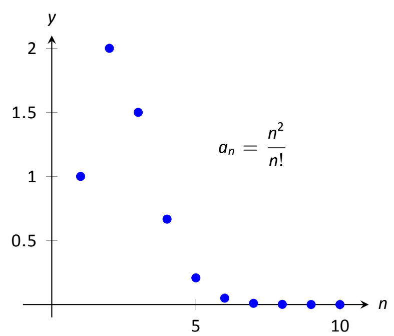

Figure 8.8: A plot of $\{a_n\} = \{n^2/n!\}$ in Example 234.

4. Again, the plot in Figure 8.8 shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. We perform the usual analysis to confirm this.

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2}{(n+1)!} - \frac{n^2}{n!} \\ &= \frac{(n+1)^2 - n^2(n+1)}{(n+1)!} \\ &= \frac{-n^3 + 2n + 1}{(n+1)!} \end{aligned}$$

When $n = 1$, the above expression is > 0; for $n \geq 2$, the above expression is < 0. Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.

Knowing that a sequence is monotonic can be useful. In particular, if we know that a sequence is bounded and monotonic, we can conclude it converges! Consider, for example, a sequence that is monotonically decreasing and is bounded below. We know the sequence is always getting smaller, but that there is a

Notes:

bound to how small it can become. This is enough to prove that the sequence will converge, as stated in the following theorem.

Theorem 59 Bounded Monotonic Sequences are Convergent

1. Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim_{n\to\infty} a_n$ exists.
2. Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above. Then $\{a_n\}$ converges.
3. Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.

Consider once again the sequence $\{a_n\} = \{1/n\}$. It is easy to show it is monotonically decreasing and that it is always positive (i.e., bounded below by 0). Therefore we can conclude by Theorem 59 that the sequence converges. We already knew this by other means, but in the following section this theorem will become very useful.

Sequences are a great source of mathematical inquiry. The On-Line Encyclopedia of Integer Sequences (`http://oeis.org`) contains thousands of sequences and their formulae. (As of this writing, there are 218,626 sequences in the database.) Perusing this database quickly demonstrates that a single sequence can represent several different "real life" phenomena.

Interesting as this is, our interest actually lies elsewhere. We are more interested in the *sum* of a sequence. That is, given a sequence $\{a_n\}$, we are very interested in $a_1 + a_2 + a_3 + \cdots$. Of course, one might immediately counter with "Doesn't this just add up to infinity?" Many times, yes, but there are many important cases where the answer is no. This is the topic of *series*, which we begin to investigate in the next section.

Notes:

Exercises 8.1

Terms and Concepts

1. Use your own words to define a *sequence.*
2. The domain of a sequence is the ________ numbers.
3. Use your own words to describe the *range* of a sequence.
4. Describe what it means for a sequence to be *bounded*.

Problems

In Exercises 5 – 8, give the first five terms of the given sequence.

5. $\{a_n\} = \left\{\dfrac{4^n}{(n+1)!}\right\}$
6. $\{b_n\} = \left\{\left(-\dfrac{3}{2}\right)^n\right\}$
7. $\{c_n\} = \left\{-\dfrac{n^{n+1}}{n+2}\right\}$
8. $\{d_n\} = \left\{\dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)\right\}$

In Exercises 9 – 12, determine the n^{th} term of the given sequence.

9. $4, 7, 10, 13, 16, \ldots$
10. $3, -\dfrac{3}{2}, \dfrac{3}{4}, -\dfrac{3}{8}, \ldots$
11. $10, 20, 40, 80, 160, \ldots$
12. $1, 1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$

In Exercises 13 – 16, use the following information to determine the limit of the given sequences.

- $\{a_n\} = \left\{\dfrac{2^n - 20}{2^n}\right\}$; $\quad \lim\limits_{n\to\infty} a_n = 1$
- $\{b_n\} = \left\{\left(1+\dfrac{2}{n}\right)^n\right\}$; $\quad \lim\limits_{n\to\infty} b_n = e^2$
- $\{c_n\} = \{\sin(3/n)\}$; $\quad \lim\limits_{n\to\infty} c_n = 0$

13. $\{a_n\} = \left\{\dfrac{2^n - 20}{7\cdot 2^n}\right\}$
14. $\{a_n\} = \{3b_n - a_n\}$
15. $\{a_n\} = \left\{\sin(3/n)\left(1+\dfrac{2}{n}\right)^n\right\}$
16. $\{a_n\} = \left\{\left(1+\dfrac{2}{n}\right)^{2n}\right\}$

In Exercises 17 – 28, determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

17. $\{a_n\} = \left\{(-1)^n\dfrac{n}{n+1}\right\}$
18. $\{a_n\} = \left\{\dfrac{4n^2 - n + 5}{3n^2+1}\right\}$
19. $\{a_n\} = \left\{\dfrac{4^n}{5^n}\right\}$
20. $\{a_n\} = \left\{\dfrac{n-1}{n} - \dfrac{n}{n-1}\right\}, n \geq 2$
21. $\{a_n\} = \{\ln(n)\}$
22. $\{a_n\} = \left\{\dfrac{3n}{\sqrt{n^2+1}}\right\}$
23. $\{a_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$
24. $\{a_n\} = \left\{5 - \dfrac{1}{n}\right\}$
25. $\{a_n\} = \left\{\dfrac{(-1)^{n+1}}{n}\right\}$
26. $\{a_n\} = \left\{\dfrac{1.1^n}{n}\right\}$
27. $\{a_n\} = \left\{\dfrac{2n}{n+1}\right\}$
28. $\{a_n\} = \left\{(-1)^n\dfrac{n^2}{2^n - 1}\right\}$

In Exercises 29 – 34, determine whether the sequence is bounded, bounded above, bounded below, or none of the above.

29. $\{a_n\} = \{\sin n\}$
30. $\{a_n\} = \{\tan n\}$
31. $\{a_n\} = \left\{(-1)^n\dfrac{3n-1}{n}\right\}$
32. $\{a_n\} = \left\{\dfrac{3n^2-1}{n}\right\}$
33. $\{a_n\} = \{n\cos n\}$
34. $\{a_n\} = \{2^n - n!\}$

In Exercises 35 – 38, determine whether the sequence is monotonically increasing or decreasing. If it is not, determine if there is an m such that it is monotonic for all $n \geq m$.

35. $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$
36. $\{a_n\} = \left\{\dfrac{n^2 - 6n + 9}{n}\right\}$
37. $\{a_n\} = \left\{(-1)^n\dfrac{1}{n^3}\right\}$
38. $\{a_n\} = \left\{\dfrac{n^2}{2^n}\right\}$
39. Prove Theorem 56; that is, use the definition of the limit of a sequence to show that if $\lim\limits_{n\to\infty}|a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.
40. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.
    (a) Show that if $a_n < b_n$ for all n, then $L \leq K$.
    (b) Give an example where $L = K$.
41. Prove the Squeeze Theorem for sequences: Let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, and let $\{c_n\}$ be such that $a_n \leq c_n \leq b_n$ for all n. Then $\lim\limits_{n\to\infty} c_n = L$

(d) $2\pi/3$

17. (a) $2\pi(\sqrt{2}-1)$
    (b) $2\pi(1-\sqrt{2}+\sinh^{-1}(1))$

Section 7.4

1. T
3. $\sqrt{2}$
5. $4/3$
7. $109/2$
9. $12/5$
11. $-\ln(2-\sqrt{3}) \approx 1.31696$
13. $\int_0^1 \sqrt{1+4x^2}\,dx$
15. $\int_0^1 \sqrt{1+\frac{1}{4x}}\,dx$
17. $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$
19. $\int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx$
21. 1.4790
23. Simpson's Rule fails, as it requires one to divide by 0. However, recognize the answer should be the same as for $y=x^2$; why?
25. Simpson's Rule fails.
27. 1.4058
29. $2\pi\int_0^1 2x\sqrt{5}\,dx = 2\pi\sqrt{5}$
31. $2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx = \pi/27(10\sqrt{10}-1)$
33. $2\pi\int_0^1 \sqrt{1-x^2}\sqrt{1+x/(1-x^2)}\,dx = 4\pi$

Section 7.5

1. In SI units, it is one joule, i.e., one Newton–meter, or kg·m/s^2·m. In Imperial Units, it is ft–lb.
3. Smaller.
5. (a) 2450 j
   (b) 1568 j
7. 735 j
9. 11,100 ft–lb
11. 125 ft–lb
13. 12.5 ft–lb
15. 7/20 j
17. 45 ft–lb
19. $953,284$ j
21. 192,767 ft–lb. Note that the tank is oriented horizontally. Let the origin be the center of one of the circular ends of the tank. Since the radius is 3.75 ft, the fluid is being pumped to $y=4.75$; thus the distance the gas travels is $h(y)=4.75-y$. A differential element of water is a rectangle, with length 20 and width $2\sqrt{3.75^2-y^2}$. Thus the force required to move that slab of gas is $F(y)=40\cdot 45.93\cdot\sqrt{3.75^2-y^2}dy$. Total work is $\int_{-3.75}^{3.75} 40\cdot 45.93\cdot(4.75-y)\sqrt{3.75^2-y^2}\,dy$. This can be evaluated without actual integration; split the integral into $\int_{-3.75}^{3.75} 40\cdot 45.93\cdot(4.75)\sqrt{3.75^2-y^2}\,dy + \int_{-3.75}^{3.75} 40\cdot 45.93\cdot(-y)\sqrt{3.75^2-y^2}\,dy$. The first integral can be evaluated as measuring half the area of a circle; the latter integral can be shown to be 0 without much difficulty. (Use substitution and realize the bounds are both 0.)
23. (a) approx. 577,000 j
    (b) approx. 399,000 j
    (c) approx 110,000 j (By volume, half of the water is between the base of the cone and a height of 3.9685 m. If one rounds this to 4 m, the work is approx 104,000 j.)
25. 617,400 j

Section 7.6

1. Answers will vary.
3. 499.2 lb
5. 6739.2 lb
7. 3920.7 lb
9. 2496 lb
11. 602.59 lb
13. (a) 2340 lb
    (b) 5625 lb
15. (a) 1597.44 lb
    (b) 3840 lb
17. (a) 56.42 lb
    (b) 135.62 lb
19. 5.1 ft

Chapter 8

Section 8.1

1. Answers will vary.
3. Answers will vary.
5. $2, \frac{8}{3}, \frac{8}{3}, \frac{32}{15}, \frac{64}{45}$
7. $\frac{1}{3}, 2, \frac{81}{5}, \frac{512}{3}, \frac{15625}{7}$
9. $a_n = 3n+1$
11. $a_n = 10\cdot 2^{n-1}$
13. $1/7$
15. 0
17. diverges
19. converges to 0
21. diverges
23. converges to e
25. converges to 0
27. converges to 2
29. bounded
31. bounded
33. neither bounded above or below
35. monotonically increasing
37. never monotonic
39. Let $\{a_n\}$ be given such that $\lim_{n\to\infty}|a_n|=0$. By the definition of the limit of a sequence, given any $\varepsilon>0$, there is a m such that for all $n>m$, $||a_n|-0|<\varepsilon$. Since $||a_n|-0|=|a_n-0|$, this directly implies that for all $n>m$, $|a_n-0|<\varepsilon$, meaning that $\lim_{n\to\infty}a_n=0$.

41. Left to reader

Section 8.2

1. Answers will vary.
3. One sequence is the sequence of terms $\{a_i\}$. The other is the sequence of n^{th} partial sums, $\{S_n\} = \{\sum_{i=1}^{n} a_i\}$.
5. F
7. (a) $1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5269}{3600}$
   (b) Plot omitted
9. (a) $1, 3, 6, 10, 15$
   (b) Plot omitted
11. (a) $\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \frac{121}{243}$
   (b) Plot omitted
13. (a) $0.1, 0.11, 0.111, 0.1111, 0.11111$
   (b) Plot omitted
15. $\lim\limits_{n\to\infty} a_n = \infty$; by Theorem 63 the series diverges.
17. $\lim\limits_{n\to\infty} a_n = 1$; by Theorem 63 the series diverges.
19. $\lim\limits_{n\to\infty} a_n = e$; by Theorem 63 the series diverges.
21. Converges
23. Converges
25. Converges
27. Converges
29. Diverges
31. (a) $S_n = \left(\frac{n(n+1)}{2}\right)^2$
   (b) Diverges
33. (a) $S_n = 5\frac{1-1/2^n}{1/2}$
   (b) Converges to 10.
35. (a) $S_n = \frac{1-(-1/3)^n}{4/3}$
   (b) Converges to $3/4$.
37. (a) With partial fractions, $a_n = \frac{3}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$. Thus $S_n = \frac{3}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$.
   (b) Converges to $9/4$
39. (a) $S_n = \ln\left(1/(n+1)\right)$
   (b) Diverges (to $-\infty$).
41. (a) $a_n = \frac{1}{n(n+3)}$; using partial fractions, the resulting telescoping sum reduces to $S_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$
   (b) Converges to $11/18$.
43. (a) With partial fractions, $a_n = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$. Thus $S_n = \frac{1}{2}\left(3/2 - \frac{1}{n} - \frac{1}{n+1}\right)$.
   (b) Converges to $3/4$.
45. (a) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of the even series is $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}$. Each term of the even series is less than the corresponding term of the odd series, giving us our result.
   (b) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of 1 plus the even series is $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2(n-1)}$. Each term of the even series is now greater than or equal to the corresponding term of the odd series, with equality only on the first term. This gives us the result.
   (c) If the odd series converges, the work done in (a) shows the even series converges also. (The sequence of the n^{th} partial sum of the even series is bounded and monotonically increasing.) Likewise, (b) shows that if the even series converges, the odd series will, too. Thus if either series converges, the other does.
   Similarly, (a) and (b) can be used to show that if either series diverges, the other does, too.
   (d) If both the even and odd series converge, then their sum would be a convergent series. This would imply that the Harmonic Series, their sum, is convergent. It is not. Hence each series diverges.

Section 8.3

1. continuous, positive and decreasing
3. The Integral Test (we do not have a continuous definition of $n!$ yet) and the Limit Comparison Test (same as above, hence we cannot take its derivative).
5. Converges
7. Diverges
9. Converges
11. Converges
13. Converges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, as $1/(n^2+3n-5) \leq 1/n^2$ for all $n > 1$.
15. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, as $1/n \leq \ln n/n$ for all $n \geq 2$.
17. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Since $n = \sqrt{n^2} > \sqrt{n^2-1}$, $1/n \leq 1/\sqrt{n^2-1}$ for all $n \geq 2$.
19. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$:
$$\frac{1}{n} = \frac{n^2}{n^3} < \frac{n^2+n+1}{n^3} < \frac{n^2+n+1}{n^3-5},$$
for all $n \geq 1$.
21. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Note that
$$\frac{n}{n^2-1} = \frac{n^2}{n^2-1} \cdot \frac{1}{n} > \frac{1}{n},$$
as $\frac{n^2}{n^2-1} > 1$, for all $n \geq 2$.
23. Converges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.
25. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.
27. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.