The Characteristic Equation

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Review

Suppose $T : R^n \rightarrow R^n$ is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is an $n \times n$ matrix.

A number λ is said to be an eigenvalue of T (or an eigenvalue of A) if there exists a non–zero vector $\mathbf{x} \in R^n$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$ (or equivalently $A\mathbf{x} = \lambda\mathbf{x}$).

If λ is an eigenvalue of T then any non–zero vector $\mathbf{x} \in R^n$ for which $T(\mathbf{x}) = \lambda\mathbf{x}$ is said to be an eigenvector of T associated with the eigenvalue λ.

The eigenspace of an eigenvalue λ is the subspace of R^n:

$$E(\lambda) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \lambda\mathbf{x}\}.$$

Since the equation $A\mathbf{x} = \lambda\mathbf{x}$ can also be written as $(A - \lambda I_n)\mathbf{x} = \mathbf{0}_n$, then

$$E(\lambda) = \{\mathbf{x} \in R^n \mid (A - \lambda I_n)\mathbf{x} = \mathbf{0}_n\}.$$

Review (continued)

If we already know that λ is an eigenvalue of A then we can find all of the eigenvectors associated with λ. We do this by finding all of the non–trivial solutions of

$$\left(A - \lambda I_n\right)\mathbf{x} = \mathbf{0}_n.$$

$E\left(\lambda\right)$ is the subspace of R^n that consists of all of these eigenvectors together with $\mathbf{0}_n$. Thus

$$E\left(\lambda\right) = \text{Null}\left(A - \lambda I_n\right).$$

How do we find the Eigenvalues?

In order for λ to be an eigenvalue of A, the equation $(A - \lambda I_n)\,\mathbf{x} = \mathbf{0}_n$ must have solutions other than $\mathbf{x} = \mathbf{0}_n$. In other words, $(A - \lambda I_n)\,\mathbf{x} = \mathbf{0}_n$ must have non–trivial solutions.

We know (by The Square Matrix Theorem) that an equation of the form $B\mathbf{x} = \mathbf{0}_n$ (where B is some $n \times n$ matrix) has non–trivial solutions if and only if $\det(B) = 0$.

Therefore $(A - \lambda I_n)\,\mathbf{x} = \mathbf{0}_n$ has non–trivial solutions if and only if

$$\det(A - \lambda I_n) = 0.$$

The above equation is called the **characteristic equation** of the matrix A. A number λ is an eigenvalue of A if and only if λ is a solution of this equation.

Example

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

Hint: To do this remember that the eigenvalues of A are the numbers, λ, that satisfy the characteristic equation

$$\det\left(A - \lambda I_2\right) = 0.$$

Example

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}.$$

Then find the eigenspace for each eigenvalue.

Example

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -1 & 4 & -4 \\ 1 & -3 & 1 \\ 1 & -2 & 0 \end{bmatrix}.$$

Hint: The characteristic equation is a cubic equation and cubic equations are usually hard to solve by hand. But luckily this one can be factored and solved by hand.
We will do this example in class but then as **homework** find the eigenspace for each of the eigenvalues.

Eigenvalues of Triangular Matrices

If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{21} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \ldots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

is an upper triangular matrix, then

$$A - \lambda I_n = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & a_{2n} \\ 0 & 0 & \ddots & \ldots \\ 0 & 0 & 0 & a_{nn} - \lambda \end{bmatrix}$$

is also an upper triangular matrix.

Eigenvalues of Triangular Matrices (continued)

We have learned that the determinant of an upper triangular matrix is the product of the entries on the main diagonal of the matrix. Thus for the matrix $A - \lambda I_n$ we have

$$\det\left(A - \lambda I_n\right) = \left(a_{11} - \lambda\right)\left(a_{22} - \lambda\right)\cdots\left(a_{nn} - \lambda\right)$$

and the characteristic equation of A is thus

$$\left(a_{11} - \lambda\right)\left(a_{22} - \lambda\right)\cdots\left(a_{nn} - \lambda\right) = 0.$$

This equation is easy to solve because it is factored and we see that the eigenvalues of A are in fact the numbers on the main diagonal of A.

Example

By quick inspection, find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & -3 & 5 \\ 0 & 2 & -5 \\ 0 & 0 & 4 \end{bmatrix}.$$

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Proof: Suppose that A is invertible. Then, by the Square Matrix Theorem, $\det(A) \neq 0$. This means that $\det(A - 0I_n) \neq 0$ and hence $\lambda = 0$ does not satisfy the characteristic equation of A. Therefore $\lambda = 0$ is not an eigenvalue of A.

Conversely, suppose that $\lambda = 0$ is not an eigenvalue of A. Then $\det(A - 0I_n) \neq 0$ and hence $\det(A) \neq 0$. By the Square Matrix Theorem, A is invertible.

Similar Matrices

An $n \times n$ matrix A is said to be **similar** to an $n \times n$ matrix B if there exists an invertible $n \times n$ matrix P such that

$$A = P^{-1}BP.$$

Note that if A is similar to B then B is also similar to A because if $A = P^{-1}BP$ then

$$B = PAP^{-1} = \left(P^{-1}\right)^{-1} A \left(P^{-1}\right).$$

It thus makes sense to say that A and B are **similar to each other**.

Example

Let A be the matrix

$$A = \begin{bmatrix} -5 & -4 \\ -1 & -3 \end{bmatrix}.$$

1. Find the eigenvalues of A.
2. Construct a matrix B that is similar to A. (*Hint:* Just choose any invertible 2×2 matrix P and let $B = P^{-1}AP$.)
3. Find the eigenvalues of B.

Theorem

If $n \times n$ matrices A and B are similar to each other, then A and B have the same eigenvalues (including multiplicities).

Proof: Suppose that B is similar to A. Then there is an invertible matrix (of the same size as A and B) such that $B = P^{-1}AP$. The characteristic equation of A is $\det(A - \lambda I_n) = 0$ and the characteristic equation of B is $\det(B - \lambda I) = 0$. However

$$\begin{aligned}
\det(B - \lambda I_n) &= \det\left(P^{-1}AP - \lambda I_n\right) \\
&= \det\left(P^{-1}AP - \lambda P^{-1}P\right) \\
&= \det\left(P^{-1}(A - \lambda I_n)P\right) \\
&= \det\left(P^{-1}\right)\det(A - \lambda I_n)\det(P) \\
&= \det(A - \lambda I_n)\det\left(P^{-1}P\right) \\
&= \det(A - \lambda I_n)\det(I) \\
&= \det(A - \lambda I_n).
\end{aligned}$$

Homework

In Section 5.2, do problems 1–17 (all) and 21–30 (all).