

Section 11.3: Fourier Cosine and Sine Series

An *eigenvalue problem* consists of a Boundary Value Problem which includes an unknown parameter λ . The task is to determine values of the parameter and corresponding nonzero functions (i.e. nontrivial) that solve the BVP. The values are called eigenvalues and the corresponding functions are called eigenfunctions.

Example: Solve

$$-u'' = \lambda u \quad \text{for} \quad 0 < x < 1$$

subject to $u(0) = 0$ and $u(1) = 0$.

We rewrote the DE as $u'' + \lambda u = 0$ whose characteristic equation is

$$m^2 + \lambda = 0.$$

We determined that if $\lambda = 0$, the only solution to the BVP is the trivial one $u(x) = 0$.

We also saw that if $\lambda < 0$ the only solution to the BVP is the trivial one $u(x) = 0$.

This tells us that there are no eigenvalues $\lambda \leq 0$. It remains to check the case $\lambda > 0$.

$$\text{If } \lambda > 0 \quad m^2 + \lambda = 0 \quad \text{let } \lambda = \alpha^2, \alpha > 0$$

$$m^2 + \alpha^2 = 0 \Rightarrow m^2 = -\alpha^2 \Rightarrow m = \pm i\alpha$$

$$u_1(x) = \cos(\alpha x) \quad \text{and} \quad u_2(x) = \sin(\alpha x)$$

$$\text{and} \quad u = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

$$\text{Apply } u(0) = 0 \quad \text{and} \quad u(1) = 0$$

$$u(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$$

$$u(1) = C_2 \sin(\alpha) = 0$$

This holds if $C_2 = 0$ or if $\sin(\alpha) = 0$

$$\sin(\alpha) = 0 \text{ if } \alpha = n\pi \text{ for } n=1, 2, 3, \dots$$

So there are infinitely many eigenvalues

$$\lambda_n = (n\pi)^2 = n^2 \pi^2$$

The associated eigen functions are $u_n(x) = \sin(n\pi x)$

A Related Eigenvalue Problem

It can be shown that the problem

$$-u'' = \lambda u \quad \text{for } 0 < x < 1 \quad \text{subject to } u'(0) = u'(1) = 0$$

has eigenvalues and eigenfunctions

$$\lambda_n = n^2 \pi^2 \quad n \geq 0, \quad u_0 = 1, \quad u_n(x) = \cos(n\pi x), \quad n \geq 1$$

Another Related Eigenvalue Problem

It can be shown that the problem

$$-u'' = \lambda u \text{ for } 0 < x < 1 \quad \text{subject to} \quad u(0) = u(1) \text{ and } u'(0) = u'(1)$$

has eigenvalues and eigenfunctions

$$\lambda_n = n^2\pi^2 \quad n \geq 0, \quad u_0(x) = 1, \quad u_n(x) = a_n \cos(n\pi x) + b_n \sin(n\pi x)$$