December 3 MATH 1113 sec. 51 Fall 2018

Section 7.5: Trigonometric Equations

In this section, we wish to consider **conditional** equations involving trigonometric functions. Our goal will be to find a solution set.

Some examples of trigonometric equations include

$$2\cos(x) - 1 = 0, \qquad \sin\theta\cos\theta + \sin\theta = 0, \qquad 2\tan^2 x - \tan x - 1 = 0,$$

$$\csc 2\theta = \sec 2\theta, \qquad \tan^2(3x) = 3, \quad \text{and so forth.}$$

We'll use trigonometric identities, our knowledge of some trig values, and inverse trigonometric functions as needed.

A General Observation

When solving more complicated trigonometric equations, we will try to rewrite the problem in the form of **one or more** equations that look like

$$\text{One Trig Function} \quad = \quad \text{One Number}$$

We typically determine solution(s) in one period, and then extend those solutions if required.

Solutions on an Indicated Interval

(a) Find all solutions of the equation $\sec^2(x) + \tan(x) = 1$ on the interval $0 \leq x < 2\pi$.

$\sec^2 x + \tan x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x + 1 + \tan x = 1$

$-1 \qquad\qquad -1$

$\tan^2 x + \tan x = 0$

Let's factor

$\tan x\,(\tan x + 1) = 0$

By the zero product property

$\tan x = 0$ or $\tan x + 1 = 0$

$\tan x = 0$ or $\tan x = -1$

$x = 0$ or $x = \pi$

reference angle is $\frac{\pi}{4}$

since $\tan\frac{\pi}{4} = 1$

we need quad II + IV

The answers are

$x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

The solution set is

$$\left\{0, \pi, \frac{3\pi}{4}, \frac{7\pi}{4}\right\}.$$

Solutions on an Indicated Interval

(b) Find all solutions of the equation $\cos^2(2x) = \frac{1}{4}$ on the interval $0 \le x < 2\pi$.

For $0 \le x < 2\pi$, $2 \cdot 0 \le 2x < 2 \cdot (2\pi)$

$$0 \le 2x < 4\pi$$

This is 2 full rotations for $2x$

$\cos^2(2x) = \frac{1}{4} \Rightarrow \cos(2x) = \frac{1}{2}$ or $\cos(2x) = \frac{-1}{2}$

The reference angle is $\frac{\pi}{3}$.

$\cos(2x) = \frac{1}{2}$ in 2 full rotations

$2x = \frac{\pi}{3}$, $2x = \frac{5\pi}{3}$, $2x = \frac{7\pi}{3}$, $2x = \frac{11\pi}{3}$

$x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, $x = \frac{7\pi}{6}$, $x = \frac{11\pi}{6}$

$\cos(2x) = -\frac{1}{2}$ 2 rotations for $2x$

$2x = \frac{2\pi}{3}$, $2x = \frac{4\pi}{3}$, $2x = \frac{8\pi}{3}$, $2x = \frac{10\pi}{3}$

$x = \frac{2\pi}{6}$, $y = \frac{4\pi}{6}$, $x = \frac{8\pi}{6}$, $x = \frac{10\pi}{6}$

The solution set is

$$\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{2\pi}{6}, \frac{4\pi}{6}, \frac{8\pi}{6}, \frac{10\pi}{6}\right\}$$

Solutions on an Indicated Interval

(c) Find all solutions of the equation $\sin^2(3x) + 3\sin(3x) - 4 = 0$ on the interval $0 \le x < 2\pi$.

For $0 \le x < 2\pi$, $3 \cdot 0 \le 3x < 3(2\pi)$

$0 \le 3x < 6\pi$

(3 full rotations)

$$\sin^2(3x) + 3\sin(3x) - 4 = 0$$

$$u^2 + 3u - 4 = 0$$

$$(u+4)(u-1) = 0$$

$$(\sin(3x) + 4)(\sin(3x) - 1) = 0$$

$\sin(3x) + 4 = 0$ or $\sin(3x) - 1 = 0$

$\sin(3x) = -4$ or $\sin(3x) = 1$

no solutions

$-1 \le \sin\theta \le 1$

The only solution in on rotation of $3x$ is

$$3x = \frac{\pi}{2}$$

In three rotations

$$3x = \frac{\pi}{2}, \quad 3x = \frac{5\pi}{2}, \quad 3x = \frac{9\pi}{2}$$

$$x = \frac{\pi}{6}, \quad x = \frac{5\pi}{6}, \quad x = \frac{9\pi}{6}$$

The solution set is

$$\left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} \right\}$$

Using Inverse Trigonometric Functions

Find all solutions of the equation $4\sin\theta = 1$. Express answers exactly in terms of the inverse sine.

$$4\sin\theta = 1$$

$$\sin\theta = \frac{1}{4}$$ not a known "nice" sine value

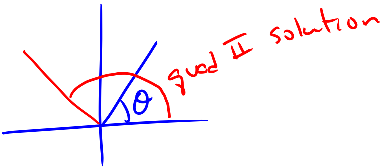
We'll find the solutions in one rotation (i.e. on $[0, 2\pi)$) then add $2\pi n$ for integers n.

One solution is $\sin^{-1}\left(\frac{1}{4}\right)$

There is a quadrant II answer.

Let $\theta = \sin^{-1}\left(\frac{1}{4}\right)$, a diagram for θ

looks like



A quad II angle with θ as its reference

From the diagram this is

$$\pi - \sin^{-1}\left(\frac{1}{4}\right)$$

The two basic solutions are

$$\sin^{-1}\left(\frac{1}{4}\right) \quad \text{and} \quad \pi - \sin^{-1}\left(\frac{1}{4}\right)$$

The solutions are

$$\theta = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n \quad \text{or}$$

$$\theta = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n \quad \text{for any integer } n$$