December 3 MATH 1113 sec. 52 Fall 2018

Section 7.5: Trigonometric Equations

In this section, we wish to consider **conditional** equations involving trigonometric functions. Our goal will be to find a solution set.

Some examples of trigonometric equations include

$$2\cos(x) - 1 = 0, \qquad \sin\theta\cos\theta + \sin\theta = 0, \qquad 2\tan^2 x - \tan x - 1 = 0,$$

$$\csc 2\theta = \sec 2\theta, \qquad \tan^2(3x) = 3, \quad \text{and so forth.}$$

We'll use trigonometric identities, our knowledge of some trig values, and inverse trigonometric functions as needed.

A General Observation

When solving more complicated trigonometric equations, we will try to rewrite the problem in the form of **one or more** equations that look like

$$\text{One Trig Function} \quad = \quad \text{One Number}$$

We typically determine solution(s) in one period, and then extend those solutions if required.

Solutions on an Indicated Interval

(a) Find all solutions of the equation $\sec^2(x) + \tan(x) = 1$ on the interval $0 \le x < 2\pi$.

$\sec^2 x + \tan x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x + 1 + \tan x = 1$

subtract 1

$\tan^2 x + \tan x = 0$

$\tan x(\tan x + 1) = 0$

By the zero product property

$\tan x = 0$ or $\tan x + 1 = 0$

$\tan x = 0$ or $\tan x = -1$

$x = 0$ or $x = \pi$

$\tan \frac{\pi}{4} = 1$, $\frac{\pi}{4}$ is the reference angle

$\tan x = -1$ in quad II and IV

we get 2 solutions

$x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

The solution set is

$$\left\{0,\ \pi,\ \frac{3\pi}{4},\ \frac{7\pi}{4}\right\}$$

Solutions on an Indicated Interval

(b) Find all solutions of the equation $\cos^2(2x) = \frac{1}{4}$ on the interval $0 \le x < 2\pi$.

We'll solve for 2x first, then divide by 2.

For $0 \le x < 2\pi$, $2\cdot 0 \le 2x < 2(2\pi)$

$0 \le 2x < 4\pi$

So 2x will be in 2 full rotations.

$\cos^2(2x) = \frac{1}{4} \Rightarrow \cos(2x) = \frac{1}{2}$ or $\cos(2x) = \frac{-1}{2}$

The reference angle will be $\frac{\pi}{3}$ for all solutions 2x.

$\cos(2x) = \frac{1}{2}$ (solutions in 2 rotations, quad I + IV)

$2x = \frac{\pi}{3}$, $2x = \frac{5\pi}{3}$, $2x = \frac{7\pi}{3}$, $2x = \frac{11\pi}{3}$

$x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, $x = \frac{7\pi}{6}$, $x = \frac{11\pi}{6}$

$\cos(2x) = \frac{-1}{2}$ (2 rotations, quads II and III)

$2x = \frac{2\pi}{3}$, $2x = \frac{4\pi}{3}$, $2x = \frac{8\pi}{3}$ $2x = \frac{10\pi}{3}$

$x = \frac{2\pi}{6}$, $x = \frac{4\pi}{6}$, $x = \frac{8\pi}{6}$, $x = \frac{10\pi}{6}$

The solution set is

$$\left\{\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6},\ \frac{2\pi}{6},\ \frac{4\pi}{6},\ \frac{8\pi}{6},\ \frac{10\pi}{6}\right\}$$

Solutions on an Indicated Interval

(c) Find all solutions of the equation $\sin^2(3x) + 3\sin(3x) - 4 = 0$ on the interval $0 \le x < 2\pi$.

If $0 \le x < 2\pi$, then

$3 \cdot 0 \le 3x < 3(2\pi)$

$0 \le 3x < 6\pi$

we want $3x$ in 3 full rotations

$\sin^2(3x) + 3\sin(3x) - 4 = 0$

$u^2 + 3u - 4 = 0$

$(u+4)(u-1) = 0$

$(\sin(3x)+4)(\sin(3x)-1)=0$

$\sin(3x)+4=0$ or $\sin(3x)-1=0$

$\sin(3x)=-4$ or $\sin(3x)=1$

$-1 \le \sin\theta \le 1$

for all real θ

no solutions

In one rotation, there is one solution

$3x = \frac{\pi}{2}$

In 3 rotations we get

$3x = \frac{\pi}{2}$, $3x = \frac{5\pi}{2}$, $3x = \frac{9\pi}{2}$

$x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, $x = \frac{9\pi}{6}$

The solution set is

$$\left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} \right\}$$

Using Inverse Trigonometric Functions

Find all solutions of the equation $4\sin\theta = 1$. Express answers exactly in terms of the inverse sine.

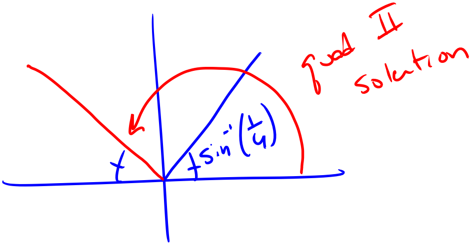
$$4\sin\theta = 1 \implies \sin\theta = \frac{1}{4}$$

We'll get the solutions in one rotation, then add $2\pi n$ for integers n.

In one rotation, there are 2 solutions.

One is $\sin^{-1}\left(\frac{1}{4}\right)$ a quadrant I answer

We need to find the quadrant II solution. The second solution has $\sin^{-1}\left(\frac{1}{4}\right)$ as its reference angle.



From the digram, the quad II answer is $\pi - \sin^{-1}\left(\frac{1}{4}\right)$

The basic solutions are

$$\sin^{-1}\left(\frac{1}{4}\right) \quad \text{and} \quad \pi - \sin^{-1}\left(\frac{1}{4}\right)$$

All solutions are given by

$$\theta = \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n, \quad \text{or}$$

$$\theta = \pi - \sin^{-1}\left(\frac{1}{4}\right) + 2\pi n$$ for n any integer