Derivatives of Polynomials without Limits

Calculus I Project

The purpose of this project is to find a general formula for the derivative of a polynomial at a point without using a limit—hence without using the power rule obtained from the definition of the derivative. (You'll essentially derive the power rule in a new way!) This can be done for polynomials by using the property that the derivative at a point gives the slope of the tangent line there. And, because the tangent line approximates a function (at least very near the point of tangency), this approach will rest on the concept of a multiple root.

Recall that c is a **root** of a function f if $f(c) = 0$. We know that if f happens to be a polynomial, then saying c is a root is the same as saying that $x - c$ is a **factor** of f. For example, 2 is a root of $f(x) = x^2 - x - 2$, and we can write $f(x) = (x-2)(x+1)$ and see that $x - 2$ is a factor of f. In general, we can say that c is a root of the polynomial $f(x)$ provided $f(x) = (x - c)q(x)$ where $q(x)$ is a polynomial. We define a double root in the following way:

Definition: Let f be a polynomial. Then c is a double root of f provided $(x - c)^2$ is a factor of f. That is, if c is double root of f, then there exists a polynomial $q(x)$ such that[1] $f(x) = (x - c)^2 q(x)$.

To proceed, we will use the fact that if $L(x) = mx + b$ is the tangent line to a polynomial $f(x)$ at some point $(c, f(c))$, then the difference

$$f(x) - L(x) \approx 0 \quad \text{for } x \text{ very close to } c.$$

If we plotted the difference $f(x) - L(x)$, it should have a flat plot near c much like the vertex of a parabola. This is because c will be a double root of this difference (note that since f is a polynomial and L is a line, the difference $f - L$ is a polynomial.) Hence we can write $f(x) - L(x) = (x - c)^2 q(x)$. This can be used to determine what the slope m of the tangent line L should be—and this we know to be $f'(c)$! Consider the following example:

Example: Let $f(x) = x^2$ and consider the point $(-1, 1)$. Let $L(x) = mx + b$ be the tangent line to the graph of f at this point (so $c = -1$). Now L passes through $(-1, 1)$, so

$$L(x) - 1 = m(x + 1) \quad \Longrightarrow \quad L(x) = m(x + 1) + 1.$$

[1] Strictly speaking, we would impose the condition that $q(c) \neq 0$, but we will relax this condition here.

The difference

$$\begin{aligned} f(x) - L(x) &= x^2 - [m(x+1)+1] \\ &= x^2 - m(x+1) - 1 \\ &= (x^2-1) - m(x+1) \\ &= (x+1)(x-1) - m(x+1) = (x+1)(x-1-m). \end{aligned}$$

Okay, so -1 must be a double root of this difference. So the second factor $(x-1-m)$ must also be $(x+1)$. Solving for m, we get

$$x - 1 - m = x + 1 \quad \Longrightarrow \quad m = -2.$$

That is, the slope of the tangent line to the graph of $f(x) = x^2$ at the point where $x = -1$ is $m = -2$. Compare this to the power rule:

$$f'(x) = 2x \quad \Longrightarrow \quad f'(-1) = 2(-1) = -2 \quad \text{BAM!}$$

Carry out the following activities.

A. Use the method above to determine the slope of the tangent line to the graph of $f(x) = x^2$ at the points

$$c = 0, 1, 3, \text{ and } -5.$$

B. Obtain a generalization for the slope of the tangent line to the graph of $f(x) = x^2$ at any point (c, c^2). (Don't just make a conjecture here, do the algebra.) Extend this to find the slope of the tangent line to the graph of $f(x) = Ax^2$ at the point (c, Ac^2) where A is any nonzero constant.

C. Now play this game with the function $f(x) = Ax^3$ where A is any nonzero constant. It might be helpful to start by taking the simple case $A = 1$ so you're just dealing with $f(x) = x^3$. You can try a few specific values of c to get a handle on the algebra involved. Then find a general formula for the slope of the tangent line to the graph of f at any point (c, Ac^3).

D. Let n be any positive integer. Find a formula for the slope of the tangent line to the graph of $f(x) = Ax^n$ at the point (c, Ac^n) using the same method. (Fortunately, there is a nice, well documented formulation for factoring $x - c$ out of the polynomial $x^n - c^n$. You can derive it yourself, or find it in a book or online.)

E. Prove (as formally as you can) the following theorem:
Theorem 1: If c is a double root of the polynomials $p(x)$ and $q(x)$, then c is a double root of the polynomial $f(x) = Ap(x) + Bq(x)$ for any choice of constants A and B.

You may assume without proof (or prove it for a little extra fun) the theorem
Theorem 2: If m_1 is the slope of the tangent line to the polynomial $p(x)$ at c and m_2 is the slope of the tangent line to the polynomial $q(x)$ at c, then the slope of the tangent line to the polynomial $f(x) = Ap(x) + Bq(x)$ at the point c is $m = Am_1 + Bm_2$.

F. Combine the results above to find a formula for the slope of the tangent line to the polynomial

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \quad \text{at the point} \quad (c, f(c)).$$

Conclude with some discussion that includes a demonstration—that is, pick a polynomial f and a value for c and demonstrate finding the equation of the tangent line using your formula. (Make f interesting. That is, don't pick a simple monomial, and choose one that is at least of degree 4.)