Exam I Review Questions: (with solutions) Math 2335 (Ritter)

Sections Covered: 1.1, 1.2, 2.2, 2.3, 3.1, 3.2, 3.3

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) (1.1 & 1.2) Write the Taylor polynomial with the remaider term for

(a) $f(x) = \sqrt{x}$ centered at $a = 4$, of order 3.

$$f(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{(x-4)^4}{128c^{7/2}}, \quad \text{for some } c \text{ between } 4 \text{ and } x.$$

(b) $f(x) = \tan^{-1}(x)$ centered at $a = 1$, of order 1.

$$f(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{c(x-1)^2}{1+c^2} \quad \text{for some } c \text{ between } 1 \text{ and } x.$$

(2) (2.2) Rearrange the function being evaluated to aviod loss of significance errors. Use algebra and function identities as needed.

(a) $f(x) = \ln(2+x) - \ln(x) = \ln\left(1 + \frac{2}{x}\right)$ $\quad x$ very large

(b) $f(x) = \sqrt[3]{8+x^2} - 2 = \dfrac{x^2}{(8+x^2)^{2/3} + 2\sqrt[3]{8+x^2} + 4}$ $\quad x \approx 0$

(c) $f(x) = \cos(\alpha) - \cos(\alpha + x) = 2\cos\left(\alpha + \frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ $\quad x \approx 0$

(3) (2.3) Find the error and the relative error in the approximations $x_A \approx x_T$.

$$\text{(a)} \quad x_T = \sqrt{2},\ x_A = 1.414, \qquad \text{(b)} \quad x_T = \ln 2,\ x_A = 0.7$$

(a) $\text{Err}(x_A) = 0.0002136$, $\text{Rel}(x_A) = 0.0001510$ $\qquad$ (b) $\text{Err}(x_A) = -0.007$, $\text{Rel}(x_A) = -0.01$

(4) (2.3, This is from problem #5 pg. 62.) In the following function evaluations $f(x_A)$, assume that the numbers x_A are correctly rounded to the number of digits shown. Bound the error $f(x_T) - f(x_A)$ and the relative error in $f(x_A)$.

$$\text{(a)} \quad \cos(1.473), \quad \text{(b)} \quad e^{2.653}$$

(a) $|E| \leq 0.0005 \sin(1.4735) \approx 0.000498$ and $|\text{Rel}(f(x_A))| \leq \sec(1.4735)|E| \approx 0.00512$

(b) $|E| \leq 0.0005 e^{2.6535} \approx 0.00710$ and $|\text{Rel}(f(x_A))| \leq e^{-2.6525}|E| \approx 0.00050$

(5) (3.1, Problem #11 pg. 78) Let α be the smallest positive root of $f(x) = 1 - x + \sin x$. Find an interval $[a, b]$ containing α. Estimate the number of iterations needed to find α using the bisection method within an accuracy of 10^{-8}.

The number of iterations will depend on the interval chosen. $f(0) = 1 > 0$ and $f(\pi) = 1 - \pi < 0$. For $[a, b] = [0, \pi]$, bisection requires 29 iterations.

(6) (3.2 & 3.3) Give Newton's method for finding $\sqrt[m]{a}$. That is, define an appropriate function whose root would be $\sqrt[m]{a}$, and set up the Newton's iteration. What would the iteration be for the Secant method?

$$f(x) = x^m - a, \quad \text{for Newton's method} \quad x_{n+1} = \frac{(m-1)x_n^m + a}{m x_n^{m-1}}$$

$$\text{for the Secant method} \quad x_{n+1} = x_n - (x_n^m - a)\frac{x_n - x_{n-1}}{x_n^m - x_{n-1}^m}, \quad n \geq 1$$

(7) (3.2) Use the inequality for convergence $|\alpha - x_0| < \left|\frac{2f'(\alpha)}{f''(\alpha)}\right|$ to find an interval for the initial guess x_0 so that Newton's method will converge for $f(x) = \sec^{-1} x - \frac{\pi}{3}$. Note, the root is $\alpha = 2$, and you can assume that $x > 0$.

$$|2 - x_0| < \frac{12}{7} \quad \Longrightarrow \quad \frac{2}{7} < x_0 < \frac{26}{7}$$

(8) (3.3) Show that if the Secant method is used to try to find the root $\alpha = 0$ of $f(x) = \sqrt[3]{x}$, that

$$x_{n+1} = -\left(x_n^{2/3} x_{n-1}^{1/3} + x_n^{1/3} x_{n-1}^{2/3}\right).$$

Will the secant method converge for any choice of x_0 and x_1 both different from zero?

Derive the formula for the secant method formula in general. It will converge. In fact, it will converge in one step if $x_0 = -x_1 \neq 0$.

(9) (3.3) Construct an algorithm to approximate $\sqrt{7}$ using the secant method. That is, define an appropriate function, and determine the secant method iteration formula. Choose a pair of initial guesses x_0 and x_1 and find the next two terms x_2 and x_3 in a sequence of approximations.

$$f(x) = x^2 - 7, \quad x_{n+1} = \frac{x_n x_{n-1} + 7}{x_n + x_{n-1}}$$

Taking $x_0 = 2$ and $x_1 = 3$ given $x_2 = \frac{13}{5}$ and $x_3 = \frac{37}{14}$.