Exam 1 Math 2254H sec. 015H

Spring 2015

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
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7	
8	
9	
10	

INSTRUCTIONS: There are 10 problems worth 10 points each. No use of notes, books, or calculator is allowed. Illicit use of notes, a book, calculator, or any smart device will result in a grade of zero on this exam. Show all of your work on the paper provided to receive full credit.

(1) $f(x) = 2 - \log_4(x)$ Find the equation of the line tangent to the graph of f at its x-intercept. Leave your final answer in the form $y = mx + b$.

$f(x) = 0 \Rightarrow 2 - \log_4 x = 0 \Rightarrow \log_4 x = 2 \Rightarrow x = 4^2 = 16$

$f'(x) = \frac{-1}{x \ln 4}$ so $f'(16) = \frac{-1}{16 \ln 4}$

The point is $(16, 0)$, and the slope $m = \frac{-1}{16 \ln 4}$.

$$y = \frac{-1}{16 \ln 4}(x - 16)$$

$$\boxed{y = \frac{-x}{16 \ln 4} + \frac{1}{\ln 4}}$$

(2) Find $\frac{dy}{dx}$ where $y = x^{\tan^{-1} x}$

$\ln y = \ln x^{\tan^{-1} x} = \tan^{-1} x \ln x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln x}{1 + x^2} + \frac{\tan^{-1} x}{x}$$

$$\frac{dy}{dx} = y\left(\frac{\ln x}{1 + x^2} + \frac{\tan^{-1} x}{x}\right)$$

$$= x^{\tan^{-1} x}\left(\frac{\ln x}{1 + x^2} + \frac{\tan^{-1} x}{x}\right)$$

(3) Evaluate the definite integral.

$$\int_e^6 \frac{dx}{x \ln x}$$

Let $u = \ln x$, $du = \frac{1}{x} dx$

when $x = e$, $u = 1$

$x = 6$, $u = \ln 6$

$$= \int_1^{\ln 6} \frac{du}{u} = \ln|u| \Big|_1^{\ln 6} = \ln|\ln 6| - \ln|1|$$

$$= \ln(\ln 6)$$

(4) Evaluate the limit (m and n are constant).

$$\lim_{x \to 0} \frac{\cos(mx) - \cos(nx)}{x^2} = \text{"}\frac{1-1}{0}\text{"} = \text{"}\frac{0}{0}\text{"}$$ use l'Hospital's rule

$$= \lim_{x \to 0} \frac{-m\sin(mx) + n\sin(nx)}{2x} = \text{"}\frac{0}{0}\text{"}$$ Use l'H again

$$= \lim_{x \to 0} \frac{-m^2 \cos(mx) + n^2 \cos(nx)}{2} = \frac{-m^2 + n^2}{2}$$

$$= \frac{n^2 - m^2}{2}$$

(5) Evaluate the indefinite integral.

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$$

Let $u = e^x$, $du = e^x\,dx$

$$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$= \sin^{-1} e^x + C$$

(6) Evaluate the indefinite integral.

$$\int x\csc^2 x\,dx$$

By parts: $u = x$ $du = dx$

$v = -\cot x$ $dv = \csc^2 x\,dx$

$$= -x\cot x + \int \cot x\,dx$$

$$= -x\cot x + \ln|\sin x| + C$$

(7) Evaluate the indefinite integral.

$$\int \sec^4 x \cot x \, dx = \int \frac{\sec^2 x \ \sec^2 x}{\tan x} \, dx$$

$$= \int \frac{(\tan^2 x + 1)}{\tan x} \sec^2 x \, dx \qquad u = \tan x \ , \ du = \sec^2 x \, dx$$

$$= \int \left(\tan x + \frac{1}{\tan x} \right) \sec^2 x \, dx$$

$$= \int \left(u + \frac{1}{u} \right) du = \frac{u^2}{2} + \ln|u| + C$$

$$= \frac{\tan^2 x}{2} + \ln|\tan x| + C$$

(8) The function $f(x) = \int_2^{\sqrt{x}} 2^{t^2} \, dt$ is one-to-one with inverse function f^{-1}. Find the equation of the line tangent to the graph of $f^{-1}(x)$ at its y-intercept—i.e. at the point $(0, f^{-1}(0))$.

$$0 = \int_2^{\sqrt{x}} 2^{t^2} \, dt \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4 \ , \ f^{-1}(0) = 4$$

$$f'(x) = 2^{(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{2^x}{2\sqrt{x}} \ , \ f'(4) = \frac{2^4}{2\sqrt{4}} = \frac{16}{4} = 4$$

The point is $(0, 4)$, and the slope is $m = (f^{-1})'(0) = \frac{1}{4}$

$$y - 4 = \frac{1}{4}(x - 0)$$

$$\boxed{y = \frac{1}{4}x + 4}$$

(9) Evaluate the definite integral

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5}$$

$x^2+2x+5 = (x^2+2x+1)+4$

$= (x+1)^2 + 2^2$

$$= \int_{-1}^{1} \frac{dx}{2^2+(x+1)^2}$$

$u = x+1, \quad du = dx$

when $x=-1, \quad u=0$

$x=1, \quad u=2$

$$= \int_0^2 \frac{du}{2^2+u^2} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\Big|_0^2$$

$$= \frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(0) = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

(10) Evaluate the limit.

$$\lim_{x\to 0^+} (4x+1)^{\cot(2x)} = \text{"}1^\infty\text{"}$$

$$\ln (4x+1)^{\cot(2x)} = \cot(2x)\ln(4x+1)$$

$$\lim_{x\to 0^+} \cot(2x)\ln(4x+1) = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan(2x)} = \text{"}\frac{0}{0}\text{"}$$ use l'H rule

$$= \lim_{x\to 0^+} \frac{\frac{4}{4x+1}}{2\sec^2(2x)} = \frac{\frac{4}{1}}{2(1)} = 2$$

Hence $\lim_{x\to 0^+} (4x+1)^{\cot(2x)} = e^2$