Exam 1 Math 2306 sec. 51

Fall 2015

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems. The point value is listed with the problem. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) (a) (10 points) Verify that the indicated family of functions is a solution of the given differential equation.

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0; \qquad y = c_1x^2 + c_2x$$

$y' = 2c_1 x + c_2$

$y'' = 2c_1$

$x^2 y'' - 2x y' + 2y =$

$x^2(2c_1) - 2x(2c_1 x + c_2) + 2(c_1 x^2 + c_2 x) =$

$2c_1 x^2 - 4c_1 x^2 - 2c_2 x + 2c_1 x^2 + 2c_2 x =$

$x^2\underbrace{(2c_1 - 4c_1 + 2c_1)}_{=0} + x\underbrace{(-2c_2 + 2c_2)}_{=0} = 0$ as required

(b) (10 points) Use the family of solutions given in part (a) to solve the initial value problem.

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0; \qquad y(1) = 2, \quad y'(1) = 1$$

$y = c_1 x^2 + c_2 x$

$y' = 2c_1 x + c_2$

$y(1) = c_1 + c_2 = 2$

$y'(1) = 2c_1 + c_2 = 1$

Subtract $-c_1 = 1 \Rightarrow c_1 = -1$

$c_1 + c_2 = 2 \Rightarrow c_2 = 2 - c_1 = 2 + 1 = 3$

$$\boxed{y = -x^2 + 3x}$$

(2) Solve each first order equation using any applicable technique. You may give either an implicit or an explicit solution, your choice.

(a) (10 points) $x\dfrac{dy}{dx} - y = \dfrac{x^2}{x^2+1}, \quad x > 0$ $\qquad \dfrac{dy}{dx} - \dfrac{1}{x}y = \dfrac{x}{x^2+1}$

$P(x) = \frac{-1}{x}$, $\int P(x)\,dx = -\int \frac{1}{x}\,dx = -\ln|x| = \ln x^{-1}$

$$\mu = e^{\int P(x)dx} = e^{\ln x^{-1}} = x^{-1}$$

$$\frac{d}{dx}\left[x^{-1}y\right] = \frac{x}{x^2+1}\cdot x^{-1} = \frac{1}{x^2+1}$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int \frac{1}{x^2+1}\,dx$$

$$x^{-1}y = \tan^{-1}x + C$$

$$\boxed{y = x\tan^{-1}x + Cx}$$

(b) (10 points) $\dfrac{dy}{dt} = 4ty - y = y(4t-1)$

$$\frac{dy}{y} = (4t-1)\,dt$$

$$\boxed{\ln|y| = 2t^2 - t + C}$$

an explicit solution is

$y = Ae^{2t^2-t}$ where $A = \pm e^C$

or

$A = 0$

(3) (20 points) A 50 volt electromotic force is applied to an LR-series circuit in which the inductance is 2 henry and the resistance is 10 ohms. Find the current $i(t)$ if the initial current $i(0) = 1$ A. Determine the current as $t \to \infty$.

$$L \frac{di}{dt} + Ri = E \implies 2\frac{di}{dt} + 10i = 50 \qquad i(0) = 1$$

$$\frac{di}{dt} + 5i = 25 \qquad P(t) = 5 \qquad \int P(t)\,dt = 5t$$

$$\mu = e^{5t}$$

$$\frac{d}{dt}\left[e^{5t} i\right] = 25 e^{5t}$$

$$\int \frac{d}{dt}\left[e^{5t} i\right] dt = \int 25 e^{5t}\,dt$$

$$e^{5t} i = \frac{25}{5} e^{5t} + C = 5e^{5t} + C$$

$$i = 5 + Ce^{-5t}$$

$$i(0) = 5 + Ce^{0} = 1 \implies C = 1 - 5 = -4$$

$$\boxed{i(t) = 5 - 4e^{-5t}}$$

as $t \to \infty$

$$\lim_{t\to\infty} i(t) = \lim_{t\to\infty} \left(5 - 4e^{-5t}\right) = 5 - 0 = 5$$

So $i \to 5$ A

(4) (10 points) A population experiences exponential growth so that it satisfies the IVP

$$\frac{dP}{dt} = kP, \quad P(0) = P_0$$

where $P(t)$ is the population at time t, t is in years, and P_0 is the initial population. If the population triples every four years, determine the value of k.

$\frac{dP}{P} = k\,dt \quad \Rightarrow \quad \ln P = kt + C \quad \Rightarrow \quad P = e^{kt+C} = Ae^{kt}$

$P(0) = Ae^0 = P_0 \quad \Rightarrow \quad A = P_0$

Hence $P(t) = P_0 e^{kt}$. Given $P(4) = 3P_0$

$3P_0 = P_0 e^{4k} \quad \Rightarrow \quad 3 = e^{4k}$

$\ln 3 = 4k \quad \Rightarrow \quad k = \frac{1}{4}\ln 3$

The value of k is $\frac{1}{4}\ln 3$.

(5) (10 points) Use the fact that the derivative of a constant function is zero to find all possible constant solutions, $y = c$, to the ODE.

$$\frac{dy}{dx} + 3 = y^2 + 2y$$

Let $y = c$, $\frac{dy}{dx} = 0$

$0 + 3 = y^2 + 2y$

$0 = y^2 + 2y - 3$

$0 = (y+3)(y-1)$

There are two constant solutions

$y = -3$ or $y = 1$.

(6) (20 points) A tank contains 100 gallons of water into which 10 lbs of salt is dissolved. A solution containing 1 lb of salt per gallon is being pumped in at a rate of 2 gallons per minute, and the well mixed solution is being pumped out at the same rate. Find the amount of salt $A(t)$ in the tank at time t where t is in minutes.

The incoming rate $= 2$ gal/min

Concentration $= 1 \frac{\text{lb}}{\text{gal}}$

The outgoing rate $= 2$ gal/min

Conc $= \frac{A \text{ lb}}{V \text{ gal}}$

Since rate in = rate out $V = 100$ gal @ all times

$$\frac{dA}{dt} = r_i c_i - r_o c_o = 2 \cdot 1 - 2 \cdot \frac{A}{100} = 2 - \frac{1}{50} A$$

The initial amount of salt is 10 lb, so $A(0) = 10$

$$\frac{dA}{dt} + \frac{1}{50} A = 2, \quad A(0) = 10$$

$P(t) = \frac{1}{50}$, $\int P(t)\,dt = \int \frac{1}{50}\,dt = \frac{t}{50}$ $\quad \mu = e^{\frac{t}{50}}$

$$\frac{d}{dt}\left[e^{\frac{t}{50}} A \right] = 2 e^{\frac{t}{50}}$$

$$\int \frac{d}{dt}\left[e^{\frac{t}{50}} A \right] dt = \int 2 e^{\frac{t}{50}}\,dt$$

$$e^{t/50} A = \frac{2}{1/50} e^{t/50} + C$$

$$A = 100 + C e^{-\frac{t}{50}}$$

$$A(0) = 10 = 100 + C \Rightarrow C = -90$$

$$\boxed{A(t) = 100 - 90 e^{-\frac{t}{50}}}$$