Exam 1 Math 2306 sec. 54

Fall 2015

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems. The point value is listed with the problem. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Solve each first order equation using any applicable technique. You may give either an implicit or an explicit solution, your choice.

(a) (10 points) $\dfrac{dy}{dx}+(\tan x)y = \sec x, \quad 0 < x < \dfrac{\pi}{2}$ $\quad P(x) = \tan x \quad \int P(x)dx = \ln|\sec x|$

$$\mu = e^{\ln|\sec x|} = |\sec x| = \sec x \text{ for } 0 < x < \frac{\pi}{2}$$

$$\frac{d}{dx}[\sec x\, y] = \sec x \sec x = \sec^2 x$$

$$\int \frac{d}{dx}[\sec x\, y]\,dx = \int \sec^2 x\, dx$$

$$\sec x\, y = \tan x + C$$

$$y = \frac{\tan x + C}{\sec x} = \sin x + C\cos x$$

(b) (10 points) $\dfrac{dr}{d\theta} = e^{-r}\sin\theta$ $\quad \dfrac{dr}{e^{-r}} = \sin\theta\, d\theta$

$$\int e^r dr = \int \sin\theta\, d\theta$$

$$\boxed{e^r = -\cos\theta + C}$$

an explicit soln. is $r = \ln(c - \cos\theta)$

(2) The temperature T of a volitile reaction satisfies the IVP[1]

$$\frac{dT}{dt} = T^2, \quad T(0) = T_0$$

where T is in hundreds of degrees K and t is in minutes.

(a) (8 points) Solve the equation for the initial temperature $T_0 = 3$. (Simplify your solution to avoid compound fractions.)

$$\frac{dT}{dt} = T^2, \quad T(0) = 3$$

$$\frac{1}{T^2}\, dt = dt \quad \Rightarrow \quad \int T^{-2}\, dT = \int dt$$

$$-T^{-1} = t + C \quad \Rightarrow \quad T = \frac{-1}{C+t}$$

$$T(0) = \frac{-1}{C} = 3 \Rightarrow C = \frac{-1}{3}, \quad T = \frac{-1}{-\frac{1}{3} + t} \cdot \frac{-3}{-3}$$

$$\boxed{T(t) = \frac{3}{1-3t}}$$

(b) (2 points) Show that the system *blows up*, i.e. that $T \to \infty$ as $t \to \frac{1}{T_0}^-$.

Here $\frac{1}{T_0} = \frac{1}{3}$. As $t \to \frac{1}{3}^-$, $1 - 3t \to 0^+$

so $\lim_{t \to \frac{1}{3}^-} T(t) = \lim_{t \to \frac{1}{3}^-} \frac{3}{1-3t} = \infty$

[1] This type of equation is called a **doomsday equation**.

(3) (a) (10 points) Verify that the indicated family of functions is a solution of the given differential equation.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0; \qquad y = c_1e^{-x} + c_2e^{2x}$$

$y' = -c_1 e^{-x} + 2c_2 e^{2x}$

$y'' = c_1 e^{-x} + 4c_2 e^{2x}$

$y'' - y' - 2y = c_1 e^{-x} + 4c_2 e^{2x} - (-c_1 e^{-x} + 2c_2 e^{2x}) - 2(c_1 e^{-x} + c_2 e^{2x})$

$= c_1 e^{-x} + 4c_2 e^{2x} + c_1 e^{-x} - 2c_2 e^{2x} - 2c_1 e^{-x} - 2c_2 e^{2x}$

$= e^{-x}\underbrace{(c_1 + c_1 - 2c_1)}_{=0} + e^{2x}\underbrace{(4c_2 - 2c_2 - 2c_2)}_{=0}$

$= 0$ as required

(b) (10 points) Use the family of solutions given in part (a) to solve the initial value problem.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0; \qquad y(0) = 1, \quad y'(0) = -7$$

$y = c_1 e^{-x} + c_2 e^{2x}$

$y' = -c_1 e^{-x} + 2c_2 e^{2x}$

$y(0) = c_1 + c_2 = 1$

$y'(0) = -c_1 + 2c_2 = -7$

add $3c_2 = -6$

$c_2 = -2$

$c_1 = 1 - c_2 = 1 + 2 = 3$

$$\boxed{y = 3e^{-x} - 2e^{2x}}$$

(4) (20 points) A tank originally contains 500 gallons of water into which 10 lbs of salt is dissolved. Fresh water (salt free) is pumped in at a rate of 3 gallons per minute, and the well mixed solution is pumped out at the faster rate of 5 gallons per minute. Derive an IVP for the amount of salt $A(t)$ in lbs in the tank at time t in minutes. **Do not solve this IVP.**

Originally 10lb of salt $\Longrightarrow$ $A(0) = 10$

Incoming

rate $r_i = 3 \frac{\text{gal}}{\text{min}}$

Conc. $c_i = 0 \frac{\text{lb}}{\text{gal}}$

Outgoing

rate $r_o = 5 \frac{\text{gal}}{\text{min}}$

Conc $c_o = \frac{A}{V} \frac{\text{lb}}{\text{gal}} = \frac{A}{500+3t-5t} \text{lb/gal} = \frac{A}{500-2t}$

$$\frac{dA}{dt} = 3 \cdot 0 - 5 \frac{A}{500-2t}$$

$$\boxed{\frac{dA}{dt} = \frac{-5}{500-2t} A, \quad A(0) = 10}$$

(5) (10 points) Find all values of the constant m such that $y = e^{mx}$ solves the ODE.

$$\frac{d^2y}{dx^2} = 4y$$

$y' = me^{mx}$, $y'' = m^2 e^{mx}$

$m^2 e^{mx} = 4 e^{mx}$

$\Longrightarrow$ $m^2 = 4$ $\Rightarrow$ $m = 2$ or $m = -2$.

(6) (20 points) A 50 volt electromotive force is applied to an RC-series circuit in which the resistance is 10 ohms and the capacitance is 0.1 farads. Find the charge $q(t)$ on the capacitor if the initial charge $q(0) = 0$ coulomb.

$$R \frac{dq}{dt} + \frac{1}{C} q = E \implies 10 \frac{dq}{dt} + \frac{1}{0.1} q = 50$$

In standard form

$$\frac{dq}{dt} + q = 5 \qquad q(0) = 0$$

$$P(t) = 1 \implies \int P(t)\,dt = t \implies \mu = e^{t}$$

$$\frac{d}{dt}\left[e^{t} q \right] = 5e^{t}$$

$$\int \frac{d}{dt}\left[e^{t} q \right] dt = \int 5e^{t}\,dt$$

$$e^{t} q = 5e^{t} + C$$

$$q = 5 + Ce^{-t}$$

$$q(0) = 5 + Ce^{0} = 0 \implies C = -5$$

$$\boxed{q(t) = 5 - 5e^{-t}}$$