Exam 1 Math 2306 sec. 57

Fall 2017

Name: Solution

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems; the point values are listed with the problems. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) (15 points) Characterize each first order differential equation. In particular, identify the equation as being

(a) Separable

(b) Linear

(c) Bernoulli

(d) Exact

You can identify the type using (a),...,(d). It is possible that an equation may be of more than one type. For example, if an equation is both linear and separable, "(a) and (b)" would be the correct answer. **You are not being asked to solve any of these equations.**

i. $\dfrac{dx}{dt} + xt = 0$ (a) and (b)

$\dfrac{dx}{dt} = -x \cdot t$

ii. $x^2 e^y \dfrac{dy}{dx} = \dfrac{x^4}{y}$

$\dfrac{dy}{dx} = \dfrac{x^4}{x^2} \cdot \dfrac{1}{ye^y}$ (a)

iii. $\dfrac{dy}{dx} - \dfrac{2y}{x} = x^{-1}y^{-1}$ (c)

$n = -1$

iv. $\dfrac{dy}{dt} = -(\sin t)y + \cos t$ (b)

$y' + \sin t \; y = \cos t$

v. $\dfrac{dr}{d\theta} = \dfrac{e^{\theta+2r}}{r\theta} = \dfrac{e^{\theta}}{\theta} \cdot \dfrac{e^{2r}}{r}$ (a)

(2) (10 points) (a) Determine whether the function $y = xe^{2x}$ is a solution of the differential equation

$$y'' - 4y' + 4y = 0$$

$y = xe^{2x}$

$y' = e^{2x} + 2xe^{2x}$

$y'' = 2e^{2x} + 2e^{2x} + 4xe^{2x}$

$4xe^{2x} + 4e^{2x} - 4(e^{2x} + 2xe^{2x}) + 4xe^{2x} =$

$4xe^{2x} + 4e^{2x} - 4e^{2x} - 8xe^{2x} + 4xe^{2x} =$

$xe^{2x}(4-8+4) + e^{2x}(4-4) = 0$

$0 + 0 = 0$

Yes it is a solution.

(b) (10 points) Determine whether the function $y = \ln(3x)$ is a solution of the differential equation

$$x^2y'' + xy' + y = 0$$

$y = \ln(3x)$

$y' = \frac{3}{3x} = \frac{1}{x}$

$y'' = \frac{-1}{x^2}$

$x^2\left(\frac{-1}{x^2}\right) + x\left(\frac{1}{x}\right) + \ln 3x =$

$-1 + 1 + \ln 3x = \ln 3x \neq 0$

No, it is not a solution.

(3) (5 points) Find an integrating factor $\mu(y)$ that could be used to make the differential equation exact. **You are only being asked for the integrating factor. You do not have to solve the equation.**

$y\,dx + (4x - 6y)\,dy = 0$ $\qquad \frac{\partial M}{\partial y} = 1$, $\quad \frac{\partial N}{\partial x} = 4 \neq \frac{\partial M}{\partial y}$

$\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4-1}{y} = \frac{3}{y}$ depends only on y.

$\mu = e^{\int \frac{3}{y} dy} = e^{3\ln y} = e^{\ln y^3} = y^3$

$\mu = y^3$ is an integrating factor

(4) (15 points) Solve the initial value problem. (You may provide an implicit or explicit solution, your choice.)

$$\frac{dy}{dx} = e^{-x}\cos^2 y, \quad y(0) = 0$$

$$\sec^2 y \frac{dy}{dx} = e^{-x}$$

$$\int \sec^2 y \, dy = \int e^{-x} \, dx$$

$$\tan y = -e^{-x} + C \qquad y(0) = 0$$

$$\tan 0 = -e^{0} + C$$

$$0 = -1 + C \Rightarrow C = 1$$

$$\boxed{\tan y = 1 - e^{-x}}$$

(5) (15 points) Show that the equation is exact, and solve the differential equation.

$(\cos x - y)\,dx + (e^y - x)\,dy = 0$ $\quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = -1 = \frac{\partial M}{\partial y}$ Exact.

$$F(x,y) = \int M \, dx$$

$$= \int (\cos x - y) \, dx = \sin x - xy + g(y)$$

$$\frac{\partial F}{\partial y} = -x + g'(y) = e^y - x \Rightarrow e^y = g'(y)$$

$$g(y) = e^y.$$

So the solutions are given by

$$\sin x - xy + e^y = C$$

(6) (15 points) Find the general solution of the differential equation.

$$\frac{dy}{dx} - \frac{3}{x}y = \frac{x^3}{\sqrt{1-x^2}}$$

$P(x) = \frac{-3}{x}$ $\mu = e^{\int \frac{-3}{x} dx} = e^{-3 \ln |x|} = x^{-3}$

$$(x^{-3} y)' = x^{-3} \frac{x^3}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{d}{dx}(x^{-3} y)\, dx = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$x^{-3} y = \sin^{-1} x + C$$

$$\boxed{y = x^3 \sin^{-1} x + C x^3}$$

(7) (15 points) Solve the first order differential equation.

$$\frac{dy}{dx}+\frac{2}{x}y=-3x^4y^2$$

Bernoulli. $u=y^{1-2}=y^{-1}=\frac{1}{y}$

$$-y^{-2}\frac{dy}{dx}=\frac{du}{dx} \Rightarrow \frac{dy}{dx}=-y^2\frac{du}{dx}$$

$$-y^2\frac{du}{dx}+\frac{2}{x}y=-3x^4y^2$$

$$\frac{du}{dx}-\frac{2}{x}\frac{y}{y^2}=-3x^4\frac{y^2}{-y^2}=-3x^4 \qquad \frac{y}{y^2}=\frac{1}{y}=u$$

$$\frac{du}{dx}-\frac{2}{x}u=3x^4 \qquad P(x)=\frac{-2}{x} \qquad \mu=e^{-\int\frac{2}{x}dx}=e^{-2\ln|x|}=x^{-2}$$

$$(x^{-2}u)'=3x^4\cdot x^{-2}=3x^2$$

$$\int\frac{d}{dx}(x^{-2}u)\,dx=\int 3x^2\,dx$$

$$x^{-2}u=x^3+C$$

$$u=x^5+Cx^2$$

$y=\frac{1}{u}$ so $\boxed{y=\frac{1}{x^5+Cx^2}}$