Exam 1 Math 2306 sec. 58

Spring 2016

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	

INSTRUCTIONS: There are 8 problems worth 12 points each/ You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Determine if each ODE is linear or nonlinear. If nonlinear, explain why (for example, identify at least one term in the equation that makes it nonlinear).

(a) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 4y = e^x$ The equation is linear.

(b) $\frac{dy}{dx} = \sqrt{1-y^2}$ This is nonlinear. y^2 and $\sqrt{1-y^2}$ can each be called nonlinear terms.

(c) $\left(\frac{d^2x}{dt^2}\right)^2 + 2x\frac{dx}{dt} = 0$ This is nonlinear. $\left(\frac{d^2x}{dt^2}\right)^2$ is a nonlinear term. So is $x\frac{dx}{dt}$.

(d) $\cos(t)\frac{du}{dt} + \sin(t)\,u = 1$ This is linear.

(2) Identify the independent variable, the dependent variable, and the order of each ODE.

(a) $x\frac{dx}{dt} - t^2 = xt$ Independent t Dependent x Order 1

(b) $\cos\theta\frac{d^2y}{d\theta^2} = \frac{d^3y}{d\theta^3}$ Independent θ Dependent y Order 3

(c) $\ddot{x} + 2\dot{x} + 4x = \cos t$ Independent t (time) Dependent x Order 2

(d) $\frac{d^4x}{dy^4} - xy = 1$ Independent y Dependent x Order 4

(3) Solve the initial value problem.

$$\frac{d^2y}{dx^2} = 12x^2 - 4e^{2x}, \quad y(0) = 0, \quad y'(0) = 2$$

$y' = \int y''\,dx = \int (12x^2 - 4e^{2x})\,dx = 4x^3 - 2e^{2x} + C_1$

$y'(0) = 4\cdot 0 - 2e^0 + C_1 = 2 \Rightarrow -2 + C_1 = 2 \quad C_1 = 4$

$y = \int y'\,dx = \int (4x^3 - 2e^{2x} + 4)\,dx = x^4 - e^{2x} + 4x + C_2$

$y(0) = 0 - e^0 + 4\cdot 0 + C_2 = 0 \Rightarrow -1 + C_2 = 0 \quad C_2 = 1$

The solution to the IVP is

$y = x^4 - e^{2x} + 4x + 1$

(4) Given that $y = c_1 e^x + c_2 e^{-2x}$ is a two parameter family of solutions of $y'' + y' - 2y = 0$, find the solution of the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 2, \quad y'(0) = -7$$

$y = C_1 e^x + C_2 e^{-2x}$ $\qquad y(0) = C_1 + C_2 = 2$

$y' = C_1 e^x - 2C_2 e^{-2x}$ $\qquad y'(0) = C_1 - 2C_2 = -7$

$C_1 + C_2 = 2$ $\qquad C_1 = 2 - C_2 = -1$

$-\ C_1 - 2C_2 = -7$ $\qquad C_1 = -1$

$3C_2 = 9$

$C_2 = 3$

So the soln. to the IVP is

$y = -e^x + 3e^{-2x}$.

(5) Solve the first order separable ODE. (Implicit or explicit, your choice.)

$$\frac{dy}{dx} = \frac{xe^x}{y^2} \quad \Rightarrow \quad y^2 \frac{dy}{dx} = xe^x$$

$$\int y^2\, dy = \int xe^x\, dx \qquad \begin{array}{ll} u = x & du = dx \\ v = e^x & dv = e^x\, dx \end{array}$$

$$\frac{1}{3}y^3 = xe^x - \int e^x\, dx$$

$$= xe^x - e^x + C$$

$$\boxed{\frac{1}{3}y^3 = xe^x - e^x + C}$$

is a one-parameter family of solutions defined implicitly

(6) Find the solution of the initial value problem. (Implicit or explicit, your choice.)

$$x\,dy = y\,dx \qquad y(1) = e$$

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx \quad \Rightarrow \quad \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + C \qquad \text{Applying } y(1) = e$$

$$\ln|e| = \ln 1 + C \quad \Rightarrow \quad 1 = C$$

$$\ln|y| = \ln|x| + 1$$

If we exponentiate, we can find y explicitly

$$|y| = e^{\ln|x| + 1} \quad \Rightarrow \quad y = ex$$

(7) Find all values of m such that $y = e^{mx}$ is a solution of the differential equation

$$y'' - 5y' + 6y = 0.$$

$$\left.\begin{array}{l} y = e^{mx} \\ y' = me^{mx} \\ y'' = m^2 e^{mx} \end{array}\right\} \Rightarrow \quad y'' - 5y' + 6y = m^2 e^{mx} - 5me^{mx} + 6e^{mx} = 0$$

$$e^{mx}(m^2 - 5m + 6) = 0$$

$$e^{mx}(m-2)(m-3) = 0$$

This holds if $m=2$ or if $m=3$.

Hence $y = e^{mx}$ solves the ODE if

$m=2$ or $m=3$.

(8) Verify that for any constant c, the function $y = cx\ln x - 3$ is a solution of the ODE

$$x^2y'' - xy' + y = -3$$

$$\left.\begin{array}{l} y = cx\ln x - 3 \\ y' = c\ln x + cx \cdot \frac{1}{x} = c\ln x + c \\ y'' = \frac{c}{x} \end{array}\right\} \Rightarrow \quad x^2y'' - xy' + y =$$

$$x^2\left(\frac{c}{x}\right) - x(c\ln x + c) + cx\ln x - 3 =$$

$$cx - cx\ln x - cx + cx\ln x - 3 = -3$$

$$-3 = -3$$

an identity.

Hence $y = cx\ln x - 3$ solves the

ODE for any constant c.