Exam 1 Math 2306 sec. 60

Spring 2018

Name: ____________________________

Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________________

INSTRUCTIONS: There are 7 problems; the point values are listed with the problems. You may use one sheet (8.5” × 11”) of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
<td></td>
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<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
</tbody>
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(1) (20 points) For each differential equation, identify

(a) The order
(b) the dependent variable
(c) the independent variable, and
(d) whether the equation is linear or nonlinear.

i. \( \frac{d^3 y}{dx^3} + 5x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} = e^{2x} \)

(a) 3  (b) y  (c) x  (d) linear

ii. \( \frac{dx}{dt} = \cos(t) \sin(x) \)

(a) 1  (b) x  (c) t  (d) nonlinear

iii. \( x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1)y = 0 \)

(a) 2  (b) y  (c) x  (d) linear

iv. \( \frac{d^4 u}{dv} + \left( \frac{du}{dv} \right)^5 = 1 \)

(a) 4  (b) u  (c) v  (d) nonlinear

v. \( 3 \frac{dy}{dt} = 2y - \frac{d^2 y}{dt^2} \implies y'' + 3y' - 2y = 0 \)

(a) 2  (b) y  (c) t  (d) linear
(2) (15 points) Use separation of variables to solve the differential equation. Give your final answer as an **explicit** solution.

\[ \frac{dy}{dx} = 8x^3y \]

\[ \frac{1}{y} \frac{dy}{dx} = 8x^3 \]

\[ \int \frac{1}{y} \, dy = \int 8x^3 \, dx \]

\[ \ln |y| = 2x^4 + C \]

\[ y = e^{2x^4+C} = ke^{2x^4}, \quad k = e^C \]

\[ y = ke^{2x^4} \]

(3) (15 points) Solve the initial value problem.

\[ \frac{d^2y}{dx^2} = 12x^2 - \frac{2}{x^2}, \quad y(1) = 0, \quad y'(1) = 0 \]

\[ y' = \int (12x^2 - \frac{2}{x^2}) \, dx = 4x^3 + \frac{2}{x} + C_1 \]

\[ y'(1) = 4 + 2 + C_1 = 0 \quad \Rightarrow \quad C_1 = -6 \]

\[ y = \int (4x^3 + \frac{2}{x} - 6) \, dx = x^4 + 2\ln|x| - 6x + C_2 \]

\[ y(1) = 1 + 2\ln 1 - 6 + C_2 = 0 \quad \Rightarrow \quad C_2 = 5 \]

\[ y = x^4 + 2\ln|x| - 6x + 5 \]
(4) (15 points) Solve the initial value problem.

\[
\frac{dy}{dx} - 2y = 2xe^x \quad y(1) = 0
\]

\[
P(x) = 2 \quad \mu = e^{\int P(x) \, dx} = e^{\int 2 \, dx} = e^{2x}
\]

\[
\frac{d}{dx}(e^{2x}y) = 2xe^{2x} \quad e^{-2x} \cdot y = \int 2x \, dx = x^2 + C
\]

\[
y = x^2e^x + Ce^x
\]

\[
y(1) = 1e^2 + Ce^2 = 0 \quad \Rightarrow \quad C = -1
\]

\[
y = x^2e^x - e^x
\]

(5) (10 points) Consider the differential equation. Without attempting to solve it, determine whether each statement is true or false. (Write T or F in the blank provided.)

\[
\frac{dy}{dx} = y^4 + e^y
\]

(i) If \( y \) is a solution, then \( \frac{d^2y}{dx^2} = 4y^3 + e^y. \) \( \underline{F} \)

(ii) The constant function \( y(x) = e \) is a solution. \( \underline{F} \)

(iii) Any solution \( y(x) \) must be an increasing function. \( \underline{T} \)

(iv) The differential equation is linear. \( \underline{F} \)

(v) The differential equation is separable. \( \underline{T} \)
(6) The current $y(t)$ in a certain series circuit satisfies the given initial value problem.

$$\frac{1}{2} \frac{dy}{dt} + 5y = 10 \quad y(0) = 0 \quad \Rightarrow \quad \frac{dy}{dt} + 10y = 20$$

(a) (12 points) Find the current $y(t)$ for all $t > 0$ (that is, solve the initial value problem).

$$P(t) = 10, \quad \mu = e^{\int P(t) dt} = e^{\int 10 dt} = e^{10t}$$

$$\frac{d}{dt} (e^{10t} y) = 20 e^{10t} \Rightarrow e^{10t} \frac{dy}{dt} = \int 20 e^{10t} dt$$

$$= 2 e^{10t} + C$$

$$\Rightarrow \quad y = 2 + Ce^{-10t}$$

$$y(0) = 2 + C = 0 \Rightarrow C = -2$$

$$y = 2 - 2e^{-10t}$$

(b) (3 points) Find the long time current in the circuit, $\lim_{t \to \infty} y(t)$.

$$\lim_{t \to \infty} y = \lim_{t \to \infty} (2 - 2e^{-10t}) = 2 - 0 = 2$$

(7) (10 points) Determine all numbers $m$ such that the function $y = e^{mx}$ is a solution to the differential equation.

$$y'' + 4y' + 4y = 0$$

$$y = e^{mx}$$

$$y' = me^{mx}$$

$$y'' = m^2 e^{mx}$$

$$m^2 e^{mx} + 4me^{mx} + 4e^{mx} = 0$$

$$e^{mx} (m^2 + 4m + 4) = 0$$

$$e^{mx} (m + 2)^2 = 0$$

Then there is one number $m = -2$. 