Exam 1 Math 2306 sec. 60

Fall 2018

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Classify each differential equation as being (i) first order linear, (ii) first order separable, or (iii) a Bernoulli equation.

(a) $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{x}{\sqrt{y}}$ Bernoulli $\left(n = -\frac{1}{2}\right)$

(b) $y^2\dfrac{dy}{dt} - t^2\sin(y) = 0$ $\dfrac{dy}{dx} = t^2\dfrac{\sin y}{y^2}$ Separable

(c) $e^z\cos(2t) = \dfrac{dz}{dt}$ Separable

(d) $\tan x\dfrac{dy}{dx} + \sin xy = x$ 1st order linear

2. Consider the differential equation

$$x^2y'' - xy' + y = 0.$$

Determine whether each function is a solution or is not a solution of this ODE. (Be sure to clearly state your conclusions!)

(a) $y = x^3$

$y' = 3x^2$

$y'' = 6x$

$x^2(6x) - x(3x^2) + x^3$

$6x^3 - 3x^3 + x^3 = 4x^3 \neq 0$

No   Not a solution

(b) $y = x \ln x$

$y' = \ln x + 1$

$y'' = \frac{1}{x}$

$x^2\left(\frac{1}{x}\right) - x(\ln x + 1) + x\ln x =$

$x - x\ln x - x + x\ln x = 0$

Yes   Is a solution

(c) $y = \sqrt{x}$

$y' = \frac{1}{2}x^{-1/2}$

$y'' = \frac{-1}{4}x^{-3/2}$

$x^2\left(\frac{-1}{4}x^{-3/2}\right) - x\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2} =$

$\frac{-1}{4}x^{1/2} - \frac{1}{2}x^{1/2} + x^{\frac{1}{2}} = \frac{1}{4}x^{1/2} \neq 0$

No   Not a solution

(d) $y = 2x + \frac{1}{x}$

$y' = 2 - \frac{1}{x^2}$

$y'' = \frac{2}{x^3}$

$x^2\left(\frac{2}{x^3}\right) - x\left(2 - \frac{1}{x^2}\right) + 2x + \frac{1}{x} =$

$\frac{2}{x} - 2x + \frac{2}{x} + 2x + \frac{1}{x} =$

$\frac{5}{x} \neq 0$

No   Not a solution

3. Solve the initial value problem. Your answer can be implicit or explicit, your choice.

$$\frac{dr}{d\theta} = \frac{1}{r^2\theta} \qquad r(1) = 2$$

separable

$$\int r^2 \, dr = \int \frac{1}{\theta} \, d\theta \Rightarrow \frac{1}{3} r^3 = \ln|\theta| + C$$

$$r^3 = 3\ln|\theta| + k \quad \text{where } k = 3C$$

Applying the IC

$$8 = 3\ln(1) + k \Rightarrow k = 8 \quad \text{as } \ln 1 = 0$$

The solution is given implicitly by

$$r^3 = 3\ln|\theta| + 8$$

4. Find the solution to the initial value problem. The general solution of the ODE is given (**you do not need to verify it**).

$$y'' - y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 1 \qquad \text{gen. sol.} \quad y = c_1 e^{3x} + c_2 e^{-2x}$$

$$\left. \begin{aligned} y &= c_1 e^{3x} + c_2 e^{-2x} \\ y' &= 3c_1 e^{3x} - 2c_2 e^{-2x} \end{aligned} \right\} \Rightarrow \begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= 3c_1 - 2c_2 = 1 \end{aligned}$$

$$c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

$$3(-c_2) - 2c = 1 \Rightarrow -5c_2 = 1 \qquad c_2 = \frac{-1}{5}, \quad c_1 = \frac{1}{5}$$

The solution is

$$y = \frac{1}{5} e^{3x} - \frac{1}{5} e^{-2x}$$

5. Solve the first order Bernoulli equation.

$$\frac{dy}{dx} + y = xe^{-2x}y^{-1}$$

$u = y^{1-(-1)} = y^2$ $\quad \frac{du}{dx} = 2y\frac{dy}{dx}$

$y\frac{dy}{dx} + y^2 = xe^{-2x} \Rightarrow \frac{1}{2}\frac{du}{dx} + u = xe^{-2x}$

$\frac{du}{dx} + 2u = 2xe^{-2x}$ $\quad P(x) = 2$ so $\mu = e^{\int 2dx} = e^{2x}$

$$\frac{d}{dx}\left(e^{2x}u\right) = 2xe^{-2x}e^{2x} = 2x$$

$$\int \frac{d}{dx}\left(e^{2x}u\right)dx = \int 2x\,dx$$

$$e^{2x}u = x^2 + C$$

$$u = x^2e^{-2x} + Ce^{-2x}.$$

Since $y^2 = u$, $\quad y = \sqrt{u}$.

The solution is

$$\boxed{y = \sqrt{x^2e^{-2x} + Ce^{-2x}}}$$

6. Solve the initial value problem. Your answer should be explicit. The domain of the solution is given.

$$\sec x \frac{dy}{dx} + y = 1, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}, \qquad y(0) = 0$$

Standard form $\frac{dy}{dx} + \frac{1}{\sec x} y = \frac{1}{\sec x}$

$\frac{dy}{dx} + \cos x \, y = \cos x \qquad P(x) = \cos x$

The integrating factor $\mu = e^{\int \cos x \, dx} = e^{\sin x}$

$\frac{d}{dx}\left(e^{\sin x} y\right) = \cos x \, e^{\sin x}$

$\int \frac{d}{dx}\left(e^{\sin x} y\right) dx = \int \cos x \, e^{\sin x} dx$

$e^{\sin x} y = e^{\sin x} + C$

If $u = \sin x$ then $du = \cos x \, dx$

The integral is $\int e^u du$

$y = 1 + C e^{-\sin x}$

Applying the I.C.

$y(0) = 0 = 1 + Ce^{-\sin 0} = 1 + Ce^0 = 1 + C$

$C = -1$

$$\boxed{\text{The solution is } \; y = 1 - e^{-\sin x}}$$

The following may or may not be useful: $\cos(0) = 1, \quad \sin(0) = 0, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \sin\left(\frac{\pi}{2}\right) = 1.$