Exam 1 Math 3260 sec. 51

Spring 2020

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use a calculator with matrix capabilities. No wifi or 4G enabled device can substitute for a calculator. **No use of a text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

1. The matrices A and B shown below are row equivalent. Use this to find the solution set of the given system of equations. Give your answer in parametric vector form.

$$A = \begin{bmatrix} 3 & -6 & 1 & 6 & 14 \\ -1 & 2 & 0 & -1 & -4 \\ 2 & -4 & 1 & 5 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} 3x_1 & - & 6x_2 & + & x_3 & + & 6x_4 & = & 14 \\ -x_1 & + & 2x_2 & & & - & x_4 & = & -4 \\ 2x_1 & - & 4x_2 & + & x_3 & + & 5x_4 & = & 10 \end{array}$$

$x_1 = 4 + 2x_2 - x_4$

$x_3 = 2 - 3x_4$

x_2, x_4 - free

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

2. Determine all values of the variable h such that the given matrix is the augmented matrix of a consistent linear system.

(a) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & h & 5 \end{bmatrix}$

$-3R_1 + R_2 \to R_2$ $\begin{bmatrix} 1 & 3 & 4 \\ 0 & h-9 & -7 \end{bmatrix}$

must be a pivot position

To be consistent, we require $h - 9 \neq 0$,

So all h except 9 (i.e. $h \neq 9$)

(b) $\begin{bmatrix} 1 & 4 & h \\ 3 & 12 & 5 \end{bmatrix}$

$-3R_1 + R_2 \to R_2$ $\begin{bmatrix} 1 & 4 & h \\ 0 & 0 & 5-3h \end{bmatrix}$

needs to be zero.

To be consistent, we require $5 - 3h = 0$,

So $h = \frac{5}{3}$

3. Suppose the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ has two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$. Show that if $\mathbf{w}$ is any vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{w}$ is also a solution of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.

Suppose $\vec{w}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$. Then

$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some scalars c_1 and c_2.

Now, $A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{0}$. Note that

$$A\vec{w} = A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2$$
$$= c_1\vec{0} + c_2\vec{0} = \vec{0}$$

That is, $A\vec{w} = \vec{0}$. Hence $\vec{w}$ solves the homogeneous system $A\vec{x} = \vec{0}$.

4. Consider the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

(a) Explain why no computations are necessary to conclude that the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ must be linearly dependent.

There are 4 vectors with only 3 entries each.
4 vectors in $\mathbb{R}^3$ are lin. dependent.

(b) Find a linear dependence relation for the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

$$[\vec{a}_1 \; \vec{a}_2 \; \vec{a}_3 \; \vec{a}_4] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \vec{a}_4 = -\vec{a}_1 + \vec{a}_3$$

A relation is

$$\vec{a}_1 - \vec{a}_3 + \vec{a}_4 = \vec{0}$$

(c) Consider the 3×4 matrix $A = [\mathbf{a}_1 \; \mathbf{a}_2 \; \mathbf{a}_3 \; \mathbf{a}_4]$. Characterize the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. (Parametric, or parametric vector form, your choice.)

From the above $[A \; \vec{0}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

$x_1 = x_4$
$x_2 = 0$
$x_3 = -x_4$
x_4 - free

In vector form, $A\vec{x} = \vec{0}$ if

$$\vec{x} = x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

5. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation. Suppose

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad T(\mathbf{u}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

(a) Evaluate each of

(i) $T(3\mathbf{u}) = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

(ii) $T(\mathbf{u} - 2\mathbf{v}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$

(iii) $T\left(\frac{1}{2}(\mathbf{u} + \mathbf{v})\right) = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(iv) $T\left(\frac{1}{2}(\mathbf{u} - \mathbf{v})\right) = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

(b) Use the fact that $\mathbf{e}_1 = \frac{1}{2}(\mathbf{u} + \mathbf{v})$ and $\mathbf{e}_2 = \frac{1}{2}(\mathbf{u} - \mathbf{v})$ to find the standard matrix for T.

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

using (iii) and (iv) above

$A = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

6. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(a) Show that every vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ is in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$. (For every pair of numbers a and b.)

Let $\vec{b} = \begin{bmatrix} a \\ b \end{bmatrix}$

$[\vec{u}\ \vec{v}\ \vec{b}] = \begin{bmatrix} 1 & 2 & a \\ 3 & -1 & b \end{bmatrix}$ $-3R_1 + R_2 \Rightarrow R_2$

$\begin{bmatrix} 1 & 2 & a \\ 0 & -7 & b-3a \end{bmatrix}$ From this ref column 1 and 2 are pivot columns.

Since the last column is not a pivot column $x_1\vec{u} + x_2\vec{v} = \vec{b}$ is always consistent.

(b) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ have standard matrix $A = [\mathbf{u}\ \mathbf{v}]$. (The vectors $\mathbf{u}$ and $\mathbf{v}$ are the columns of A.)

(i) Is T onto? (Justify)

By the above $A\vec{x} = \vec{b}$ is always consistent. Hence T is onto.

(ii) Is T one to one? (Justify)

Consider $A\vec{x} = \vec{0}$

$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $x_1 = x_2 = 0$

Only the trivial solution exists, hence T is one to one.