Exam 1 Math 3260 sec. 55

Spring 2020

Name: (4 points) ____________________

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use a calculator with matrix capabilities. No wifi or 4G enabled device can substitute for a calculator. **No use of a text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

1. The matrices A and B shown below are row equivalent. Use this to find the solution set of the given system of equations. Give your answer in parametric vector form.

$$A = \begin{bmatrix} -2 & 2 & -3 & -2 & -8 \\ 3 & -3 & 3 & 1 & 10 \\ 2 & -2 & 2 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} -2x_1 & + & 2x_2 & - & 3x_3 & - & 2x_4 & = & -8 \\ 3x_1 & - & 3x_2 & + & 3x_3 & + & x_4 & = & 10 \\ 2x_1 & - & 2x_2 & + & 2x_3 & & & = & 4 \end{array}$$

$x_1 = 6 + x_2$

$x_3 = -4$

$x_4 = 4$

x_2 - free

$$\vec{x} = \begin{bmatrix} 6 \\ 0 \\ -4 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2. Determine all values of the variable h such that the given matrix is the augmented matrix of a consistent linear system.

(a) $\begin{bmatrix} -1 & 3 & 2 \\ 2 & h & 1 \end{bmatrix}$

$2R_1 + R_2 \to R_2$ $\begin{bmatrix} -1 & 3 & 2 \\ 0 & h+6 & 5 \end{bmatrix}$

needs to be a pivot position

The system is consistent as long as $h+6 \neq 0$.

So for all $h \neq -6$.

(b) $\begin{bmatrix} 1 & -3 & h \\ 4 & -12 & 10 \end{bmatrix}$

$-4R_1 + R_2 \to R_2$ $\begin{bmatrix} 1 & -3 & h \\ 0 & 0 & 10-4h \end{bmatrix}$

needs to not be a pivot

The system is consistent if $10-4h = 0$.

So $h = \frac{10}{4} = \frac{5}{2}$

3. Suppose the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ has two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$. Show that if $\mathbf{w}$ is any vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{w}$ is also a solution of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.

Let $\vec{w}$ be in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$. Then $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some scalars c_1, c_2. We know that

$A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{0}$.

Note that $A\vec{w} = A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2$

$= c_1 \vec{0} + c_2 \vec{0} = \vec{0}$

That is, $A\vec{w} = \vec{0}$. Hence $\vec{w}$ solves $A\vec{x} = \vec{0}$.

4. Consider the vectors

$$\mathbf{a}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

(a) Explain why no computations are necessary to conclude that the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ must be linearly dependent.

There are four vectors in $\mathbb{R}^3$. We have more vectors than entries in each vector.

(b) Find a linear dependence relation for the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\vec{a}_3 = -\vec{a}_1 + \vec{a}_2$

A relation is $\vec{a}_1 - \vec{a}_2 + \vec{a}_3 = \vec{0}$

(c) Consider the 3×4 matrix $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4]$. Characterize the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. (Parametric, or parametric vector form, your choice.)

Using (b) above $[A \ \vec{0}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$x_1 = x_3$
$x_2 = -x_3$ $\quad x_3$ - free
$x_4 = 0$

Solutions $\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

5. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation. Suppose

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad T(\mathbf{u}) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$$

(a) Evaluate each of

(i) $T(3\mathbf{u}) = 3\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$

(ii) $T(\mathbf{u} - 2\mathbf{v}) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$

(iii) $T\left(\frac{1}{2}(\mathbf{u} + \mathbf{v})\right) = \frac{1}{2}\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -8 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 8 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

(iv) $T\left(\frac{1}{2}(\mathbf{u} - \mathbf{v})\right) = \frac{1}{2}\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -8 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

(b) Use the fact that $\mathbf{e}_1 = \frac{1}{2}(\mathbf{u} + \mathbf{v})$ and $\mathbf{e}_2 = \frac{1}{2}(\mathbf{u} - \mathbf{v})$ to find the standard matrix for T.

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

using (iii) and (iv)

$$A = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$$

6. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$.

(a) Show that every vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$. (For every pair of numbers a and b.)

Let $\vec{b} = \begin{bmatrix} a \\ b \end{bmatrix}$ Consider $x_1\vec{u} + x_2\vec{v} = \vec{b}$

$$\begin{bmatrix} 1 & 4 & a \\ 2 & -2 & b \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 4 & a \\ 0 & -10 & b-2a \end{bmatrix}$$

Clearly a pivot position

We see from the ref that columns 1 and 2 are pivot columns.

Hence the equation is always consistent.

(b) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ have standard matrix $A = [\mathbf{u}\ \mathbf{v}]$. (The vectors $\mathbf{u}$ and $\mathbf{v}$ are the columns of A.)

(i) Is T onto? (Justify) Yes. By the above $A\vec{x} = \vec{b}$ is always consistent.

(ii) Is T one to one? (Justify) Consider $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ There is only the trivial solution.

Hence T is one to one.