Exam 2 Math 2254H sec. 015H

Spring 2015

Name: 4 points Solutions

Your signature (required) confirms that you agree to practice academic honesty.

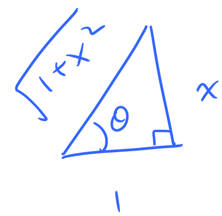
Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. No use of notes, books, or calculator is allowed. Illicit use of notes, a book, calculator, or any smart device will result in a grade of zero on this exam and may result in a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Evaluate the integral using any applicable method.

$$\int \frac{dx}{(1+x^2)^{3/2}}$$



$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$\sqrt{1+x^2} = \sec\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{\sec^3\theta}$$

$$= \int \frac{1}{\sec\theta} \, d\theta$$

$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + C$$

$$= \frac{x}{\sqrt{1+x^2}} + C$$

(2) For each rational function, determine the form of the partial fraction decomposition. **It is NOT necessary to find any of the coefficients A, B, etc.**

(a) $\dfrac{x}{(x^2-16)(x^2+16)} = \dfrac{x}{(x-4)(x+4)(x^2+16)}$

$$= \frac{A}{x-4} + \frac{B}{x+4} + \frac{Cx+D}{x^2+16}$$

(b) $\dfrac{2x+1}{x^3-5x} = \dfrac{2x+1}{x(x^2-5)} = \dfrac{2x+1}{x(x-\sqrt{5})(x+\sqrt{5})}$

$$= \frac{A}{x} + \frac{B}{x-\sqrt{5}} + \frac{C}{x+\sqrt{5}}$$

(c) $\dfrac{x^2-2}{x^3(x^2+x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+x+1}$

x^2+x+1 is irreducible

(3) Determine if the improper integral is convergent or divergent. If convergent, evaluate it.

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= \lim_{t\to\infty} -2e^{-\sqrt{x}} \Big|_1^t$$

$$= \lim_{t\to\infty} \left(-2e^{-\sqrt{t}} + 2e^{-\sqrt{1}}\right)$$

$$= 0 + 2e^{-1}$$

$$= \frac{2}{e}$$ it is convergent

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \int 2e^{-u}\,du$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2du = \frac{1}{\sqrt{x}}\,dx$

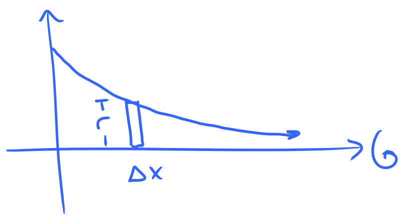
$$= -2e^{-u} + C$$

$$= -2e^{-\sqrt{x}} + C$$

(4) The region bounded between the curves

$$y = 0, \quad \text{and} \quad y = \frac{1}{\sqrt{1+x^2}}$$

on the interval $0 \leq x < \infty$ is rotated about the x-axis to form a solid. This solid happens to have a finite volume. Find the volume of this solid.



$r = y = \frac{1}{\sqrt{1+x^2}}$

A disk of thickness Δx has volume

$$V_D = \pi r^2 \Delta x = \pi \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \Delta x$$

$$= \frac{\pi}{1+x^2} \Delta x$$

Summing from $x=0$ to $x \to \infty$, the volume is

$$V = \int_0^\infty \frac{\pi}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{\pi}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \pi \tan^{-1} x \Big|_0^t$$

$$= \lim_{t\to\infty} \left(\pi \tan^{-1} t - \pi \tan^{-1} 0\right)$$

$$= \pi\left(\frac{\pi}{2}\right) - 0$$

$$= \frac{\pi^2}{2}$$

(5) Evaluate the indefinite integral.

$$\int \frac{x^3 + x^2 + 3x + 1}{x^2 + x - 2}\, dx$$

$$\begin{array}{r} x \\ x^2+x-2 \overline{\big)\, x^3 + x^2 + 3x + 1} \\ -\,(x^3 + x^2 - 2x) \\ \hline 5x + 1 \end{array}$$

$$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \implies$$

$$5x+1 = A(x-1) + B(x+2)$$

$x = -2 \qquad -9 = -3A \implies A = 3$

$x = 1 \qquad 6 = 3B \implies B = 2$

$$\int \frac{x^3 + x^2 + 3x + 1}{x^2 + x - 2}\, dx = \int \left(x + \frac{3}{x+2} + \frac{2}{x-1} \right) dx$$

$$= \frac{x^2}{2} + 3\ln|x+2| + 2\ln|x-1| + C$$

(6) (a) Deteremine if the integral is convergent or divergent. If convergent, evaluate it.

$$\int_1^2 \frac{dx}{x-1} = \lim_{t \to 1^+} \int_t^2 \frac{dx}{x-1}$$

$$= \lim_{t \to 1^+} \ln|x-1| \Big|_t^2$$

$$= \lim_{t \to 1^+} (\ln|1| - \ln|t-1|) = \infty$$

The integral is divergent.

(b) Based on your findings above, argue whether the following integral is convergent or divergent. Justify your conclusion. **Note:** It is not possible (or necessary) to find an antiderivative for this integrand.

$$\int_1^2 \frac{e^x}{x-1}\, dx$$

For $1 \leq x \leq 2$ $\quad e^1 \leq e^x \leq e^2$.

Since $x-1 \geq 0$ for $x \geq 1$

$$\frac{e}{x-1} < \frac{e^x}{x-1} < \frac{e^2}{x-1}$$

$e\int_1^2 \frac{dx}{x-1}$ is infinite. Since $\frac{e^x}{x-1}$ is larger than $\frac{e}{x-1}$, $\int_1^2 \frac{e^x}{x-1}\,dx$ must also be infinite – i.e. the integral diverges.