

# Exam 2 Math 2306 sec. 51

Fall 2015

Name: \_\_\_\_\_ *Solutions*

Your signature (required) confirms that you agree to practice academic honesty.

Signature: \_\_\_\_\_

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

**No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

(1) Consider the homogeneous differential equation for which one solution is given.

$$4x^2y'' + y = 0; \quad y_1 = x^{1/2} \ln x$$

(a) Find a second linearly independent solution  $y_2$ .

$$y'' + \frac{1}{4x^2}y = 0 \quad P(x) = 0$$

$$\begin{aligned} u &= \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx \\ &= \int \frac{1}{(x^{1/2} \ln x)^2} dx = \int \frac{(ln x)^{-2}}{x} dx \\ &\quad v = \ln x, \quad dv = \frac{1}{x} dx \\ &= \int v^{-2} dv \\ &= -\frac{1}{v} = -\frac{1}{\ln x}, \quad y_2 = y_1 u = x^{1/2} \ln x \cdot -\frac{1}{\ln x} = -x^{-1/2} \\ &\boxed{y_2 = -\sqrt{x}} \quad (\text{the sign may be dropped}) \end{aligned}$$

(b) Solve the initial value problem  $4x^2y'' + y = 0, \quad y(1) = -2, \quad y'(1) = 0$ .

$$y = C_1 x^{1/2} \ln x + C_2 x^{-1/2}$$

$$y' = \frac{1}{2} C_1 x^{-1/2} \ln x + C_1 \frac{x^{1/2}}{x} + \frac{1}{2} C_2 x^{-1/2}$$

$$y(1) = C_1 \ln 1 + C_2 1 = -2 \Rightarrow C_2 = -2$$

$$y'(1) = \frac{1}{2} C_1 \ln 1 + C_1 1 + \frac{1}{2} C_2 1 = 0$$

$$C_1 = -\frac{1}{2} C_2 = 1$$

$$\boxed{y = x^{1/2} \ln x - 2x^{-1/2}}$$

(2) Find the general solution of each differential equation.

$$(a) \quad y'' - 12y' + 36y = 0 \quad m^2 - 12m + 36 = 0 \quad (m-6)^2 = 0$$

$m = 6$  repeated root

$$y_1 = e^{6x}, \quad y_2 = xe^{6x}$$

$$\left| y = C_1 e^{6x} + C_2 x e^{6x} \right.$$

$$(b) \quad y'' + 2y' + 10y = 0 \quad m^2 + 2m + 10 = 0$$

$$m^2 + 2m + 1 + 9 = 0$$

$$(m+1)^2 = -9 \quad m+1 = \pm 3i$$

$$m = -1 \pm 3i$$

$$\alpha = -1, \beta = 3$$

$$y_1 = e^{-x} \cos(3x), \quad y_2 = e^{-x} \sin(3x)$$

$$\left| y = C_1 e^{-x} \cos(3x) + C_2 e^{-x} \sin(3x) \right.$$

(3) For each nonhomogeneous DE, determine the *form* of the particular solution when using the method of undetermined coefficients. **Do not solve for any coefficients A, B, etc.** (You may wish to refer to results of problem (2).)

$$(a) \quad y'' - 12y' + 36y = x^2 - 4e^{6x} \quad y_c = C_1 e^{6x} + C_2 x e^{6x} \quad (2a)$$

$$y_{p_1} = Ax^2 + Bx + C \quad (\text{this is okay})$$

$$y_{p_2} = D e^{6x} \rightarrow \text{solves the homogeneous eqn}$$

$$y_{p_2} = Dx^2 e^{6x} \quad \text{this is correct}$$

$$\boxed{y_p = Ax^2 + Bx + C + Dx^2 e^{6x}}$$

$$(b) \quad y'' + 2y' + 10y = 7x \cos(4x) \quad y_c = C_1 e^{-x} \cos(3x) + C_2 e^{-x} \sin(3x) \quad (2b)$$

$$\boxed{y_p = (Ax + B) \cos(4x) + (Cx + D) \sin(4x)}$$

This works as is.

$$(c) \quad y'' + 2y' + 10y = e^{-x} \cos(3x) \quad \text{Same } y_c \text{ as above}$$

$$y_p = A e^{-x} \cos(3x) + B e^{-x} \sin(3x) \rightarrow \text{won't work, it is } y_c$$

Correct  
it

$$\boxed{y_p = Ax e^{-x} \cos(3x) + Bx e^{-x} \sin(3x)}$$

(4) Find the general solution of the nonhomogeneous differential equation.

$$y'' + 2y' - 3y = 9x$$

Get  $y_c$ :  $m^2 + 2m - 3 = 0 \quad (m+3)(m-1) = 0$   
 $m_1 = -3, \quad m_2 = 1$

$$y_c = C_1 e^{-3x} + C_2 e^x$$

Get  $y_p$ : assume  $y_p = Ax + B$

$$y_p' = A$$

$$y_p'' = 0$$

$$y_p'' + 2y_p' - 3y_p = 9x$$

$$0 + 2A - 3(Ax + B) = 9x$$

$$-3Ax + (2A - 3B) = 9x$$

$$-3A = 9 \Rightarrow A = -3$$

$$2A - 3B = 0 \Rightarrow B = \frac{2}{3}A = -2$$

$$y_p = -3x - 2$$

$$y = C_1 e^{-3x} + C_2 e^x - 3x - 2$$

(5) For each set of functions, determine whether they are linearly dependent or independent on the indicated interval. (Clearly state your conclusion with justification.)

$$(a) \quad f_1(x) = xe^{2x}, \quad f_2(x) = -e^{2x}, \quad (-\infty, \infty)$$

$$W(f_1, f_2)(x) = \begin{vmatrix} xe^{2x} & -e^{2x} \\ e^{2x} + 2xe^{2x} & -2e^{2x} \end{vmatrix}$$

$$= -2xe^{4x} + e^{4x} + 2xe^{4x} = e^{4x} \neq 0$$

$W \neq 0$ , hence they are

linearly independent

$$(b) \quad y_1 = x^3, \quad y_2 = x^3 - 2x, \quad y_3 = 4x, \quad (0, \infty)$$

$$\text{Note } y_1 - y_2 - \frac{1}{2}y_3 = x^3 - (x^3 - 2x) - \frac{1}{2}(4x)$$

$$= x^3 - x^3 + 2x - 2x = 0$$

for all  $x$  in  $(0, \infty)$

So taking  $c_1 = 1, c_2 = -1, c_3 = \frac{-1}{2}$  which are not all zero

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0 \text{ for all } x \text{ in I}$$

Hence they are linearly dependent,