Exam 2 Math 2306 sec. 52

Summer 2016

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) A 200 gallon aquarium is initially filled with water into which 40 lbs of salt is dissolved. Brine containing 1 lb/gal is pumped in at a rate of 4 gal/min, and the well mixed solution is pumped out at the same rate. Determine the amount of salt $A(t)$ in lbs for all $t > 0$ (in minutes). Show that in the long run (i.e. as $t \to \infty$), the concentration of salt in the tank matches the 1 lb/gal concentration of the brine coming in.

$V(0) = 200$ gal $\qquad r_i = 4 \frac{\text{gal}}{\text{min}} = r_o \qquad c_o = \frac{A}{V} = \frac{A}{200 + (4-4)t} = \frac{A}{200}$

$c_i = 1 \frac{\text{lb}}{\text{gal}}$

$$\frac{dA}{dt} = r_i c_i - r_o c_o = 1 \frac{\text{lb}}{\text{gal}} \cdot 4 \frac{\text{gal}}{\text{min}} - 4 \frac{\text{gal}}{\text{min}} \cdot \frac{A}{200} \frac{\text{lb}}{\text{gal}}$$

$$\Rightarrow \quad \frac{dA}{dt} = 4 - \frac{4}{200} A \qquad A(0) = 40 \text{ lb}$$

Solve the IVP $\quad \frac{dA}{dt} + \frac{1}{50} A = 4, \quad A(0) = 40$

$$\mu = e^{\int \frac{1}{50} dt} = e^{\frac{1}{50} t} \qquad \frac{d}{dt}\left[e^{\frac{1}{50} t} A\right] = 4 e^{\frac{1}{50} t}$$

$$e^{\frac{1}{50} t} A = 200 e^{\frac{1}{50} t} + C \quad \Rightarrow \quad A = 200 + C e^{-\frac{1}{50} t}$$

$$A(0) = 40 = 200 + C \quad \Rightarrow \quad C = -160$$

$$\boxed{\text{so} \quad A(t) = 200 - 160 e^{-\frac{1}{50} t}}$$

As $t \to \infty \quad e^{-\frac{1}{50} t} \to 0$ so $\lim_{t \to \infty} A(t) = 200$

The volume $V = 200$ gal for all time.

So the concentration C

$$C \to \frac{200 \text{ lb}}{200 \text{ gal}} = 1 \frac{\text{lb}}{\text{gal}} \quad \text{as } t \to \infty.$$

(2) Find the general solution of the differential equation for which one solution is given.

$$x^2y'' - 2xy' + (x^2+2)y = 0, \quad y_1 = x\cos x$$

Standard form $\quad y'' - \frac{2}{x}y' + \frac{x^2+2}{x^2}y = 0$

$y_2 = y_1 u$ when $\quad u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$

Here, $P(x) = \frac{-2}{x}$ so $\quad -\int P(x)dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2$

$$u = \int \frac{e^{\ln x^2}}{(x\cos x)^2}\,dx = \int \frac{x^2}{x^2\cos^2 x}\,dx = \int \sec^2 x\,dx$$

$$= \tan x$$

$$y_2 = y_1 u = x\cos x \cdot \tan x = x\cos x\left(\frac{\sin x}{\cos x}\right) = x\sin x$$

$$\boxed{\begin{array}{l}\text{The general solution is therefore} \\ \quad y = c_1 x\cos x + c_2 x\sin x\end{array}}$$

(3) (a) Verify that the set of functions form a **fundamental solution set** for the given differential equation.

$$x^2y'' - xy' + y = 0, \quad \text{for} \quad x > 0; \quad y_1 = x, \quad y_2 = x\ln(x)$$

There are two functions for this 2^{nd} order ODE.

$y_1 = x$ $\qquad x^2y_1'' - xy_1' + y_1 \stackrel{?}{=} 0$

$y_1' = 1$ $\qquad x^2(0) - x(1) + x \stackrel{?}{=} 0$

$y_1'' = 0$ $\qquad -x + x = 0$ $\quad y_1$ Solves the ODE

$y_2 = x\ln x$ $\qquad x^2y_2'' - xy_2' + y_2 \stackrel{?}{=} 0$

$y_2' = \ln x + \frac{x}{x}$ $\qquad x^2\left(\frac{1}{x}\right) - x(\ln x + 1) + x\ln x \stackrel{?}{=} 0$

$y_2'' = \frac{1}{x}$ $\qquad x - x\ln x - x + x\ln x \stackrel{?}{=} 0$

$0 = 0$ $\quad y_2$ also solves the ODE

Compute the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix}$$

$$= x(\ln x + 1) - x\ln x = x \qquad W(y_1, y_2)(x) = x \neq 0$$

There are 2 linearly independent solutions.
Hence y_1, y_2 form a fund. soln. set.

(b) Write the general solution of the ODE.

$$y = c_1 x + c_2 x\ln x$$

(4) Find the general solution of each linear, constant coefficient, homogeneous equation.

(a) $y'' - 64y = 0$

$m^2 - 64 = 0 \Rightarrow m = \pm 8$ 2 real roots

$y_1 = e^{8x}$, $y_2 = e^{-8x}$

$$\boxed{y = C_1 e^{8x} + C_2 e^{-8x}}$$

(b) $y'' - 2y' + 10y = 0$

$m^2 - 2m + 10 = 0$

$m^2 - 2m + 1 + 9 = 0 \Rightarrow (m-1)^2 = -9$

$m - 1 = \pm 3i$

$m = 1 \pm 3i$

$\alpha = 1 \quad \beta = 3$

$y_1 = e^x \cos(3x)$, $y_2 = e^x \sin(3x)$

$$\boxed{y = C_1 e^x \cos(3x) + C_2 e^x \sin(3x)}$$

(c) $y'' + 10y' + 25y = 0$

$m^2 + 10m + 25 = 0 \Rightarrow (m+5)^2 = 0$

$m = -5$ repeated root.

$y_1 = e^{-5x}$ $\quad y_2 = xe^{-5x}$

$$\boxed{y = C_1 e^{-5x} + C_2 xe^{-5x}}$$

(5) Solve the initial value problem $y'' - 6y' + 8y = 0, \quad y(0) = 0, \quad y'(0) = -4$.

Char. Eqn $m^2 - 6m + 8 = 0 \Rightarrow (m-2)(m-4) = 0$ $m_1 = 2,\ m_2 = 4$

$y_1 = e^{2x},\ y_2 = e^{4x}$

Gen. Soln. $y = c_1 e^{2x} + c_2 e^{4x}$

$y' = 2c_1 e^{2x} + 4c_2 e^{4x}$

$y(0) = c_1 e^0 + c_2 e^0 = 0 \Longrightarrow c_1 + c_2 = 0$

$y'(0) = 2c_1 e^0 + 4c_2 e^0 = -4 \Rightarrow 2c_1 + 4c_2 = -4$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $c_1 = -c_2$

$-2c_2 + 4c_2 = -4$ $2c_2 = -4$

$c_2 = -2$

$c_1 = 2$

$\boxed{\text{The solution is } y = 2e^{2x} - 2e^{4x}}$

(6) A 250 volt battery is attached to an RC-series circuit with a resistance of 50 ohms and a capacitance of $\frac{1}{1000}$ farads. If the initial charge $q(0) = 0$, determine the charge on the capacitor for all $t > 0$.

$R\frac{dq}{dt} + \frac{1}{C}q = E \Rightarrow 50q' + \frac{1}{1/1000}q = 250$

Standard form $q' + 20q = 5$ $q(0) = 0$ $\mu = e^{\int 20\,dt} = e^{20t}$

$\int \frac{d}{dt}\left[e^{20t} q\right] dt = \int 5e^{20t}\,dt$

$e^{20t} q = \frac{5}{20} e^{20t} + k \Rightarrow q = \frac{1}{4} + k e^{-20t}$

$q(0) = \frac{1}{4} + k e^0 = 0 \Rightarrow k = \frac{-1}{4}$

$\boxed{\text{The charge is } q(t) = \frac{1}{4} - \frac{1}{4} e^{-20t}}$