Exam 2 Math 2306 sec. 53

Fall 2018

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas. **No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

(1) Use the Wronskian to determine whether each set of functions is linearly dependent or linearly independent on the interval $(0, \infty)$.

(a) $f_1(x) = \ln x, \quad f_2(x) = x \ln x$

$$W(f_1, f_2)(x) = \begin{vmatrix} \ln x & x \ln x \\ \frac{1}{x} & \ln x + 1 \end{vmatrix} = (\ln x)^2 + \ln x - \ln x = (\ln x)^2 \neq 0$$

They are independent.

(a) $g_1(x) = e^{-x}, \quad g_2(x) = e^{-x+2}$

$$W(g_1, g_2)(x) = \begin{vmatrix} e^{-x} & e^{-x+2} \\ -e^{-x} & -e^{-x+2} \end{vmatrix} = -e^{-2x+2} + e^{-2x+2} = 0$$

They are dependent.

(2) Consider the second order linear equation $x^2y'' + xy' - 9y = 0$.

(a) The function $y_1 = x^{-3}$ is a solution (you can check, but it's not necessary). Explain why $y = c_1x^{-3}$ is NOT the general solution.

The equation is 2^{nd} order, There must be 2 solutions y_1 and y_2 for general solution

$$y = C_1 y_1 + C_2 y_2$$

(b) Find the general solution.

Reduction of order $\quad y'' + \frac{1}{x}y' - \frac{9}{x^2}y = 0$

$$u = \int \frac{e^{-\int \frac{1}{x}dx}}{(x^{-3})^2}\,dx = \int \frac{e^{-\ln x}}{x^{-6}}\,dx = \int \frac{x^{-1}}{x^{-6}}\,dx = \int x^5\,dx$$

$$= \frac{x^6}{6}$$

So $y_2 = x^{-3} \cdot \frac{x^6}{6} = \frac{x^3}{6}$

Ignoring the $\frac{1}{6}$, the general solution is

$$y = C_1 x^{-3} + C_2 x^3$$

(3) Verify that the differential equation is exact, and find the general solution.
$(3x^2y + e^x)\,dx + (x^3 - \sin y)\,dy = 0$

$M_y = 3x^2 \qquad N_x = 3x^2 \Rightarrow M_y = N_x \quad$ so $F_x = M,\ F_y = N$

$$F(x,y) = \int (3x^2y + e^x)\,dx = x^3y + e^x + g(y)$$

$$\frac{\partial F}{\partial y} = x^3 + g'(y) = x^3 - \sin y \Rightarrow g'(y) = -\sin y$$

$$g(y) = \cos y$$

So $F(x,y) = x^3y + e^x + \cos y$.

The solutions are

$$\boxed{x^3y + e^x + \cos y = C}$$

(4) Verify the pair $y_1 = e^x$ and $y_2 = e^{2x}$ are a fundamental solution set for $y'' - 3y' + 2y = 0$.

There are 2 of them

$y_1 = e^x$ $\quad y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x = 0$ $\quad y_1$ solves it.

$y_1' = e^x$

$y_1'' = e^x$ $\quad y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$ $\quad y_2$ solves it too.

$y_2' = 2e^{2x}$ $\quad W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$

$y_2'' = 4e^{2x}$

They are linearly independent.

$\Longrightarrow$ They are a fundamental solution set.

(5) **(a)** Verify that $y_{p_1} = 2$ solves the nonhomogeneous equation $y'' - 3y' + 2y = 4$.

$y_{p_1}' = 0$ $\quad y_{p_1}'' - 3y_{p_1}' + 2y_{p_1} = 4$

$y_{p_1}'' = 0$ $\quad 0 - 3\cdot 0 + 2(2) = 4$ ✓

y_{p_1} solves the ODE

(b) Verify that $y_{p_2} = \frac{1}{2}e^{-x}$ solves the nonhomogenous equation $y'' - 3y' + 2y = 3e^{-x}$.

$y_{p_2}' = -\frac{1}{2}e^{-x}$ $\quad y_{p_2}'' - 3y_{p_2}' + 2y_{p_2} = 3e^{-x}$

$y_{p_2}'' = \frac{1}{2}e^{-x}$ $\quad \frac{1}{2}e^{-x} + \frac{3}{2}e^{-x} + \frac{2}{2}e^{-x} = 3e^{-x}$ $\quad y_{p_2}$ does solve the ODE

$\frac{6}{2}e^{-x} = 3e^{-x}$ ✓

(c) Use (a) and (b) together with the problem (4) from this page to write the general solution of the linear, nonhomogeneous equation

$$y'' - 3y' + 2y = 2 + 3e^{-x}$$

$y_c = c_1 y_1 + c_2 y_2$ and $y_p = y_{p_1} + y_{p_2}$

The general solution is

$$\boxed{y = c_1 e^x + c_2 e^{2x} + 2 + \frac{1}{2}e^{-x}}$$

(6) A tank originally contains 100 gallons of water into which 10 lbs of salt is dissolved. A solution containing 1 lb of salt per gallon is pumped into the tank at a rate of 2 gallons per minute, and the well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds in the tank at time t in minutes.

$$\frac{dA}{dt} = r_i c_i - r_o c_o \qquad r_i = 2 \text{ g/min} \qquad r_o = 2 \text{ gal/min}$$

$$c_i = 1 \text{ lb/gal} \qquad c_o = \frac{A}{100}$$

$$\frac{dA}{dt} = 2\cdot 1 - 2\frac{A}{100} = 2 - \frac{1}{50}A$$

$$A(0) = 10$$

$$\frac{dA}{dt} + \frac{1}{50}A = 2 \qquad p(t) = \frac{1}{50} \qquad \mu = e^{\int \frac{1}{50}dt} = e^{\frac{1}{50}t}$$

$$\left(e^{\frac{1}{50}t}A\right)' = 2e^{\frac{1}{50}t}$$

$$e^{\frac{1}{50}t}A = \int 2e^{\frac{1}{50}t}\,dt = 100e^{\frac{1}{50}t} + C$$

$$A = 100 + Ce^{-\frac{1}{50}t}$$

$$A(0) = 10 = 100 + C \Rightarrow C = -90$$

$$\boxed{A(t) = 100 - 90e^{-\frac{1}{50}t}}$$