Exam 2 Math 2306 sec. 53

Fall 2018

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

1. Let a be a constant, and consider the pair of functions $y_1 = ax^2$ and $y_2 = x^2 + 1$ defined in the interval $(0, \infty)$.

(a) Find the Wronskian of y_1 and y_2.

$$W(y_1, y_2)(x) = \begin{vmatrix} ax^2 & x^2+1 \\ 2ax & 2x \end{vmatrix} = 2ax^3 - (2ax^3 + 2ax)$$
$$= -2ax$$

$$W(y_1, y_2)(x) = -2ax$$

(b) For what values of a is the pair of functions linearly independent? (Justify)

The are independent provided $W \neq 0$.

This holds for all $a \neq 0$.

2. Find the general solution of the second order, linear differential equation. One solution is provided. Assume $x > 0$.

$$x^2y'' + 3xy' + y = 0, \qquad y_1 = x^{-1}$$

Standard form

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0$$

$$P(x) = \frac{3}{x}$$

$y_2 = u y_1$

where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}dx$ $\qquad e^{-\int P(x)dx} = e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$

$$u = \int \frac{x^{-3}}{(x^{-1})^2}dx = \int \frac{x^2}{x^3}dx = \int \frac{1}{x}dx = \ln x$$

$$y_2 = x^{-1}\ln x$$

The general solution is

$$y = c_1 x^{-1} + c_2 x^{-1}\ln x$$

3. A tank initially contains 100 liters of water into which 5 kg of salt is dissolved. Fresh water is pumped into the tank at a rate of 1 L/min. The well mixed solution is pumped out at the faster rate of 2 L/min. Determine the amount of salt $A(t)$ in kg in the tank at time t in minutes.

$V(0) = 100\text{ L}$ $\quad A(0) = 5$ kg

$r_i = 1$ L/min $\quad r_o = 2 \frac{L}{min}$

so $V(t) = 100 + (1-2)t = 100 - t$ L

$c_i = 0$ fresh water

$c_o = \frac{A}{V} = \frac{A}{100-t}$

$\frac{dA}{dt} + r_o c_o = r_i c_i \qquad \frac{dA}{dt} + \frac{2A}{100-t} = 0$

Separating $\quad \frac{dA}{dt} = \frac{-2}{100-t} A$

$$\int \frac{1}{A}\, dA = -\int \frac{2}{100-t}\, dt$$

$$\ln A = 2 \ln |100-t| + C = \ln (100-t)^2 + C$$

$$A = e^{\ln (100-t)^2 + C} = k\,(100-t)^2 \qquad k = e^C$$

$$A(0) = 5 = k\,(100)^2 \Rightarrow k = \frac{5}{100^2} = \frac{5}{10{,}000}$$

The amount of salt

$$A(t) = \frac{5}{10000}(100-t)^2 \qquad \text{for } 0 < t < 100$$

4. Consider the nonhomogeneous, linear differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 18x^2 + 4x, \quad \text{for} \quad 0 < x < \infty$$

(a) Verify that $y_{p_1} = 2x^3$ is a solution of $\quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 18x^2.$

$y_{p_1}' = 6x^2$

$x(12x) + (6x^2) \stackrel{?}{=} 18x^2$

y_{p_1} is a solution

$y_{p_1}'' = 12x$

$18x^2 = 18x^2$

true

(b) Verify that $y_{p_2} = x^2$ is a solution of $\quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 4x.$

$y_{p_2}' = 2x$

$x(2) + (2x) \stackrel{?}{=} 4x$

y_{p_2} is a solution

$y_{p_2}'' = 2$

$4x = 4x$

true

(c) Given that the general solution of $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ is $y = c_1 + c_2 \ln x$, find the solution of the initial value problem

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 18x^2 + 4x, \quad y(1) = 1, \quad y'(1) = 0$$

$y = y_c + y_p$ and $y_p = y_{p_1} + y_{p_2}$

$y = c_1 + c_2 \ln x + 2x^3 + x^2$

$y(1) = c_1 + c_2 \ln 1 + 2 \cdot 1^3 + 1^2 = 1$

$c_1 + 3 = 1$

$c_1 = -2$

$y' = \frac{c_1}{x} + 6x^2 + 2x$

$y'(1) = \frac{c_2}{1} + 6(1)^2 + 2 \cdot 1 = 0$

$c_2 + 8 = 0$

$c_2 = -8$

The solution to the IVP is

$$y = -2 - 8\ln x + 2x^3 + x^2$$

5. Solve the initial value problem

$$y\,dx + \left(2x - \frac{\sin y}{y}\right)dy = 0 \qquad y(1) = 1$$

$\frac{\partial M}{\partial y} = 1$ $\qquad \frac{\partial N}{\partial x} = 2$ $\qquad$ not exact

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-1}{2x - \frac{\sin y}{y}}$ $\qquad$ not of x alone

$\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{1}{y}$ ✓ $\qquad \mu = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

$$y^2 dx + (2xy - \sin y)\,dy = 0$$

Solution $F(x,y) = C$ $\qquad F(x,y) = \int y^2 dx = \int (2xy - \sin y)\,dy$

$F(x,y) = xy^2 + g(y)$ $\qquad \frac{\partial F}{\partial y} = 2xy + g'(y) = 2xy - \sin y$

$g'(y) = -\sin y$

$g(y) = \cos y$

So $F(x,y) = xy^2 + \cos y$

$x^2 y + \cos y = C$ $\qquad y(1) = 1$

$1^2 \cdot 1 + \cos 1 = C \implies C = 1 + \cos 1$

The solution to the IVP is

$$\boxed{xy^2 + \cos y = 1 + \cos 1}$$