Exam 2 Math 2306 sec. 54

Fall 2018

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Let a be a constant, and consider the pair of functions $y_1 = \frac{a}{x}$ and $y_2 = 1 + \frac{1}{x}$ defined in the interval $(0, \infty)$.

(a) Find the Wronskian of y_1 and y_2.

$$W(y_1,y_2)(x) = \begin{vmatrix} \frac{a}{x} & 1+\frac{1}{x} \\ \frac{-a}{x^2} & \frac{-1}{x^2} \end{vmatrix} = \frac{-a}{x^3} - \left(\frac{-a}{x^2} - \frac{a}{x^3}\right)$$

$$= \frac{a}{x^2}$$

$$W(y_1,y_2)(x) = \frac{a}{x^2}$$

(b) For what values of a is the pair of functions linearly independent? (Justify)

They're linearly independent if $W \neq 0$.
This is true for all $a \neq 0$.

2. Find the general solution of the second order, linear differential equation. One solution is provided. Assume $x > 0$.

$$x^2y'' + 5y' + 4y = 0, \qquad y_1 = x^{-2}$$

This ODE contains a typo.
It should read
$x^2y'' + 5xy' + 4y = 0$

Standard form

$y'' + \frac{5}{x}y' + \frac{4}{x^2}y = 0 \qquad P(x) = \frac{5}{x}$

I have accounted for this error in the grading.

$y_2 = uy_1$ where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$

$e^{-\int P(x)dx} = e^{-\int \frac{5}{x}dx} = e^{-5\ln x} = x^{-5}, \quad (y_1)^2 = (x^{-2})^2 = x^{-4}$

$u = \int \frac{x^{-5}}{x^{-4}}\,dx = \int \frac{x^4}{x^5}\,dx = \int \frac{1}{x}\,dx = \ln x$

$y_2 = x^{-2}\ln x$

The general solution is

$$y = c_1x^{-2} + c_2x^{-2}\ln x$$

3. A large run off tank originally contains 600 gallons of water with 50 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces of pollution per gallon. The solution is kept well mixed, and is drained from the tank at the same rate of 3 gal/hr. Determine the amount of pollution $A(t)$, in ounces, in the tank at the time t in hours.

$V(0) = 600$ gal $\qquad A(0) = 50$ oz

$r_i = 3 \frac{\text{gal}}{\text{hr}} \qquad r_o = 3 \frac{\text{gal}}{\text{hr}}$

$c_i = 5 \frac{\text{oz}}{\text{gal}}$ $\qquad$ Note $V(t) = 600$ for all t since $r_i = r_o$

$$c_o = \frac{A}{V} = \frac{A}{600}$$

$\frac{dA}{dt} + r_o c_o = r_i c_i \qquad \frac{dA}{dt} + \frac{3}{600} A = 15$

$P(t) = \frac{1}{200} \qquad \mu = e^{\int \frac{1}{200} \cdot dt} = e^{\frac{1}{200} dt}$

$$\frac{d}{dt}\left(e^{\frac{1}{200}t} A\right) = 15\, e^{\frac{1}{200}t} \Rightarrow e^{\frac{1}{200}t} A = 15(200)\, e^{\frac{1}{200}t} + C$$

$$A = 3000 + C\, e^{\frac{-1}{200}t}$$

$$A(0) = 50 = 3000 + C \qquad C = -2950$$

The amount of pollution is

$$A = 3000 - 2950\, e^{\frac{-1}{200}t} \text{ oz}$$

4. Consider the nonhomogeneous, linear differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2 + 9x^2, \quad \text{for} \quad 0 < x < \infty$$

(a) Verify that $y_{p_1} = 2x$ is a solution of $\quad x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 2.$

$y_{p_1}' = 2$ $\qquad x(0) + (2) \stackrel{?}{=} 2$ $\qquad y_{p_1}$ is a solution

$y_{p_1}'' = 0$ $\qquad 2 = 2$

true

(b) Verify that $y_{p_2} = x^3$ is a solution of $\quad x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 9x^2.$

$y_{p_2}' = 3x^2$ $\qquad x(6x) + (3x^2) \stackrel{?}{=} 9x^2$ $\qquad y_{p_2}$ is a solution

$y_{p_2}'' = 6x$ $\qquad 9x^2 = 9x^2$

true

(c) Given that the general solution of $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$ is $y = c_1 + c_2 \ln x$, find the solution of the initial value problem

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2 + 9x^2, \quad y(1) = 1, \quad y'(1) = 0$$

$y = y_c + y_p$ $\qquad y_p = y_{p_1} + y_{p_2}$ $\qquad$ so that $\quad y = c_1 + c_2 \ln x + 2x + x^3$

$y' = \dfrac{c_2}{x} + 2 + 3x^2$ $\qquad y(1) = c_1 + c_2 \ln 1 + 2\cdot 1 + 1^3 = 1$

$c_1 + 3 = 1 \qquad c_1 = -2$

$y'(1) = \dfrac{c_2}{1} + 2 + 3\cdot 1^2 = 0 \qquad c_2 + 5 = 0$

$c_2 = -5$

The solution to the IVP is

$$y = -2 - 5\ln x + 2x + x^3$$

5. Solve the initial value problem

$$y\,dx + \left(2x - e^{y^2}\right)\,dy = 0 \qquad y(1) = 1$$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2$ not exact

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-1}{2x - e^{y^2}}$ no function of x

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1}{y}$ a function of y! $\quad \mu = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

$$y^2 dx + (2xy - ye^{y^2}) dy = 0$$

$F(x,y) = \int y^2 dx = \int (2xy - ye^{y^2})\, dy$

$= xy^2 + g(y) \qquad \frac{\partial F}{\partial y} = 2xy + g'(y) = 2xy - ye^{y^2}$

$g'(y) = -ye^{y^2}$

$g(y) = -\frac{1}{2} e^{y^2}$

$F(x,y) = xy^2 - \frac{1}{2} e^{y^2}$

The solutions are defined by $F(x,y) = C$

$y(1) = 1 \Rightarrow 1 \cdot 1^2 - \frac{1}{2} e^{1^2} = C \qquad C = 1 - \frac{1}{2} e$

The solution to the IVP satisfies

$$xy^2 - \frac{1}{2} e^{y^2} = 1 - \frac{1}{2} e$$