Exam 2 Math 2306 sec. 57

Fall 2017

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ________________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems; the point values are listed with the problems. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) An aquarium is filled with 400 gallons of water into which 80 lbs of salt is dissolved. Fresh water is pumped in at a rate of 4 gallons per minute and the well mixed solution is pumped out at the same rate. Do the following to determine the amount of salt $A(t)$ in lbs in the tank at time t in minutes.

(a) (5 pts) Identify the inflow rate, the salt concentration of the inflowing fluid, and the outflow rate of fluid:

$r_i =$ 4 gal/min $c_i =$ 0 lbs/gal $r_o =$ 4 gal/min

(b) (5 pts) Identify the initial condition $A(0) =$ 80 lbs

(c) (5 pts) Identify the variable salt concentration of the outflowing fluid $c_o =$ $\frac{A}{V} = \frac{A}{400}$ lbs/gal (Remember: It contains the dependent variable A.)

(d) (10 pts) Set up and solve the initial value problem governing the amount of salt A.

$$\frac{dA}{dt} = r_i c_i - r_o c_o = 4 \cdot 0 - 4 \cdot \frac{A}{400} \qquad A(0) = 80$$

$$\frac{dA}{dt} = \frac{-1}{100} A$$

$$\frac{1}{A} \frac{dA}{dt} = \frac{-1}{100} \Rightarrow \int \frac{1}{A}\, dA = \int \frac{-1}{100}\, dt$$

$$\ln A = \frac{-1}{100} t + C \Rightarrow A = k e^{\frac{-1}{100} t} \quad \text{where } k = e^C$$

$$A(0) = k = 80 \Rightarrow k = 80$$

The amount of salt $A = 80 e^{\frac{-1}{100} t}$ lb.

(2) (10 pts) Find the general solution of the second order differential equation for which one solution is given.

$$x^2y'' + xy' - y = 0, \quad y_1(x) = \frac{1}{x}, \qquad \text{for} \quad x > 0$$

$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$ in standard form. $P(x) = \frac{1}{x}$

$y_2 = u y_1$ where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx = \int \frac{e^{-\int \frac{1}{x} dx}}{(x^{-1})^2} dx = \int \frac{e^{-\ln x}}{x^{-2}} dx$

$u = \int \frac{x^{-1}}{x^{-2}} dx = \int x \, dx = \frac{x^2}{2}$

$y_2 = u y_1 = \frac{x^2}{2} \cdot \frac{1}{x} = \frac{1}{2} x$

The general solution $y = c_1 y_1 + c_2 y_2$. Absorbing the factor $\frac{1}{2}$ into c_2

$$\boxed{y = \frac{c_1}{x} + c_2 x}$$

(3) (10 pts) Use the Wronskian to determine whether the following functions are linearly dependent or linearly independent on $(1, \infty)$. (Be sure to clearly indicate your conclusion—dependent or independent.)

$$y_1(x) = \ln x, \quad \text{and} \quad y_2(x) = x \ln x$$

$$W(y_1, y_2)(x) = \begin{vmatrix} \ln x & x \ln x \\ \frac{1}{x} & 1 \ln x + x \cdot \frac{1}{x} \end{vmatrix} = \begin{vmatrix} \ln x & x \ln x \\ \frac{1}{x} & \ln x + 1 \end{vmatrix}$$

$= \ln x (\ln x + 1) - \frac{1}{x}(x \ln x)$

$= (\ln x)^2 + \ln x - \ln x = (\ln x)^2 \neq 0$ for x in $(1, \infty)$.

Since $W \neq 0$, y_1 and y_2 are linearly independent.

(4) Find the general solution of each second order, linear, homogeneous equation.

(a) (5pts) $y''+4y'+3y=0$

$m^2+4m+3=0$ $\quad (m+3(m+1))=0$

$m_1=-3, \quad m_2=-1$

$$\boxed{y = C_1 e^{-3x} + C_2 e^{-x}}$$

(b) (5pts) $y''-3y'=0$

$m^2-3m=0 \implies m(m-3)=0$

$m_1=0 \quad y_2=3$

$y_1=e^{0x}=1$

$y_2=e^{3x}$

$$\boxed{y = C_1 + C_2 e^{3x}}$$

(c) (5pts) $y''-4y'+4y=0$

$m^2-4m+4=0 \implies (m-2)^2=0$

$y_1=e^{2x}, \quad y_2=xe^{2x}$ $\quad m=2$ repeated

$$\boxed{y = C_1 e^{2x} + C_2 x e^{2x}}$$

(d) (5pts) $y''-4y'+5y=0$

$m^2-4m+5=0 \implies m^2-4m+4+1=0$

$y_1=e^{2x}\cos x$ $\quad (m-2)^2=-1$

$y_2=e^{2x}\sin x$ $\quad m=2\pm i$

$$\boxed{y = C_1 e^{2x}\cos x + C_2 e^{2x}\sin x}$$

(5) (10 pts) Solve the initial value problem. $y'' - y = 0,\ y(0) = 1,\ y'(0) = 2.$

Charact eqn $\quad m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

$m_1 = 1,\ m_2 = -1$

$y = C_1 e^x + C_2 e^{-x} \qquad y(0) = C_1 + C_2 = 1$

$y' = C_1 e^x - C_2 e^{-x} \qquad y'(0) = C_1 - C_2 = 2$

$\left.\begin{array}{l} C_1 + C_2 = 1 \\ C_1 - C_2 = 2 \end{array}\right\} \Rightarrow 2C_1 = 3 \qquad C_1 = \frac{3}{2}$

$C_2 = 1 - C_1 = 1 - \frac{3}{2} = \frac{-1}{2}$

$y = \frac{3}{2} e^x - \frac{1}{2} e^{-x}$ is the solution to the IVP

(6) (10 pts) An RC-series circuit has a resistance of 25 ohms, capacitance of 10^{-4} farads, and is attached to a battery that provides an oscillating input voltage of $(50\cos(\omega t) + 50)$ volts. If the initial charge on the capacitor $q(0) = 0$, write out the initial value problem governing the charge $q(t)$ at time t. Put the differential equation into standard form. **Do not solve the IVP.**

$R = 25\,\Omega \qquad C = 10^{-4}\,f \qquad E = 50\cos(\omega t) + 50$

$25 \frac{dq}{dt} + \frac{1}{10^{-4}} q = 50\cos(\omega t) + 50 \qquad \frac{10^4}{25} = \frac{100}{25} \cdot 100 = 400$

In standard form, the IVP is

$\frac{dq}{dt} + 400 q = 2\cos(\omega t) + 2, \quad q(0) = 0$

(7) Consider the second order, linear, nonhomogeneous equation

$$x^2y'' + xy' - y = 2x.$$

(a) (5 pts) Verify that $y_1 = \frac{1}{x}$ solves the associated homogeneous equation. (You may recall this from problem (2) on this exam.)

$y_1 = \frac{1}{x}$

$y_1' = \frac{-1}{x^2}$

$y_1'' = \frac{2}{x^3}$

$x^2\left(\frac{2}{x^3}\right) + x\left(\frac{-1}{x^2}\right) - \frac{1}{x} =$

$\frac{2}{x} - \frac{1}{x} - \frac{1}{x} = 0$

So y_1 solves

$x^2y_1'' + xy_1' - y_1 = 0$

(b) (5 pts) Verify that $y_p = x\ln x$ solves the nonhomogeneous equation.

$y_p = x\ln x$

$y_p' = \ln x + 1$

$y_p'' = \frac{1}{x}$

$x^2\left(\frac{1}{x}\right) + x(\ln x + 1) - x\ln x =$

$x + x\ln x + x - x\ln x = 2x$

So y_p solves

$x^2y_p'' + xy_p' - y_p = 2x$

(c) (5 pts) Jack and Diane are working together to find the general solution of the nonhomogeneous equation. Jack says that the solution looks like

$$y = y_c + y_p$$

so his claim is that the general solution is

$$y = \frac{c_1}{x} + x\ln x.$$

Diane disagrees. She thinks that $y = \frac{c_1}{x} + x\ln x$ is NOT the general solution. Who is correct? And how do you know?

Diane is correct. The equation is 2^{nd} order. So a fundamental solution set for the associated homogeneous equation must have 2 lin. independent solutions in it. y_c is missing a y_2. y_c should look like $c_1y_1 + c_2y_2$.