INSTRUCTIONS: There are 7 problems; the point values are listed with the problems. You may use one sheet (8.5” × 11”) of your own prepared notes/formulas.

No use of a calculator, textbook, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.
An aquarium is filled with 400 gallons of water into which 80 lbs of salt is dissolved. Fresh water is pumped in at a rate of 4 gallons per minute and the well mixed solution is pumped out at the same rate. Do the following to determine the amount of salt $A(t)$ in lbs in the tank at time $t$ in minutes.

(a) (5 pts) Identify the inflow rate, the salt concentration of the inflowing fluid, and the outflow rate of fluid:

$$r_i = \underline{4} \text{ gal/min} \quad c_i = \underline{0} \text{ lbs/gal} \quad r_o = \underline{4} \text{ gal/min}$$

(b) (5 pts) Identify the initial condition $A(0) = \underline{80}$ lbs

(c) (5 pts) Identify the variable salt concentration of the outflowing fluid $c_o = \frac{A - A}{V} \text{ lbs/gal}$

(Remember: It contains the dependent variable $A$.)

(d) (10 pts) Set up and solve the initial value problem governing the amount of salt $A$.

$$\frac{dA}{dt} = r_i c_i - r_o c_o = 4 \cdot 0 - 4 \cdot \frac{A}{400} \quad A(0) = 80$$

$$\frac{dA}{dt} = -\frac{1}{100} A$$

$$\frac{1}{A} \frac{dA}{dt} = -\frac{1}{100} \quad \Rightarrow \int \frac{1}{A} dA = \int -\frac{1}{100} \, dt$$

$$\ln A = -\frac{1}{100} t + C \Rightarrow A = ke^{-\frac{1}{100} t} \quad \text{when } k = e^C$$

$$A(0) = k = 80 \Rightarrow k = 80$$

The amount of salt $A = 80 e^{-\frac{1}{100} t}$ lb.
(2) (10 pts) Find the general solution of the second order differential equation for which one solution is given.

\[ x^2 y'' + xy' - y = 0, \quad y_1(x) = \frac{1}{x}, \quad \text{for} \quad x > 0 \]

\[ y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0 \quad \text{in standard form.} \quad p(x) = \frac{1}{x} \]

\[ y_2 = u y_1, \quad \text{when} \quad u = \int \frac{e^{\int p(x) \, dx}}{(y_1(x))^2} \, dx = \int \frac{e^{\int \frac{1}{x} \, dx}}{(x^{-1})^2} \, dx = \int \frac{e^{\ln x}}{x^2} \, dx \]

\[ u = \int \frac{x^{-1}}{x^2} \, dx = \int x \, dx = \frac{x^2}{2} \]

\[ y_2 = u y_1 = \frac{x^2}{2} \cdot \frac{1}{x} = \frac{1}{2} x \]

The general solution is \( y = c_1 y_1 + c_2 y_2 \). Absorbing the factor \( \frac{1}{2} \) into \( c_2 \)

\[ y = \frac{c_1}{x} + c_2 x \]

(3) (10 pts) Use the Wronskian to determine whether the following functions are linearly dependent or linearly independent on \((0, \infty)\). (Be sure to clearly indicate your conclusion—dependent or independent.)

\[ y_1(x) = \ln x, \quad \text{and} \quad y_2(x) = x \ln x \]

\[ W(y_1, y_2)(x) = \begin{vmatrix} \frac{1}{x} & x \ln x \\ \ln x + x \ln x - x \end{vmatrix} = \begin{vmatrix} \ln x & x \ln x \\ \frac{1}{x} & \ln x + 1 \end{vmatrix} \]

\[ = \ln x \left( \ln x + 1 \right) - \frac{1}{x} (x \ln x) \]

\[ = (\ln x)^2 + \ln x - \ln x = (\ln x)^2 \neq 0 \quad \text{for} \quad x \in (0, \infty) \]

Since \( W \neq 0 \), \( y_1 \) and \( y_2 \) are linearly independent.
(4) Find the general solution of each second order, linear, homogeneous equation.

(a) (5pts) \( y'' + 4y' + 3y = 0 \)

\[ m^2 + 4m + 3 = 0 \]

\( m = -3, -1 \)

\[ y = c_1 e^{-3x} + c_2 e^{-x} \]

(b) (5pts) \( y'' - 3y' = 0 \)

\[ m^2 - 3m = 0 \]

\( m = 0, 3 \)

\[ y_1 = e^0 = 1, \quad y_2 = e^3x \]

\[ y = c_1 + c_2 e^3x \]

(c) (5pts) \( y'' - 4y' + 4y = 0 \)

\[ m^2 - 4m + 4 = 0 \]

\( (m-2)^2 = 0 \)

\( m = 2, 2 \) (repeated)

\[ y_1 = e^{2x}, \quad y_2 = xe^{2x} \]

\[ y = c_1 e^{2x} + c_2 xe^{2x} \]

(d) (5pts) \( y'' - 4y' + 5y = 0 \)

\[ m^2 - 4m + 5 = 0 \]

\( m = 2 \pm i \)

\[ y_1 = e^{2x} \cos x \]

\[ y_2 = e^{2x} \sin x \]

\[ y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x \]
(5) (10 pts) Solve the initial value problem. \( y'' - y = 0, \ y(0) = 1, \ y'(0) = 2. \)

\[
\begin{align*}
\text{Char. eqn} & \quad m^2 - 1 = 0 \quad \Rightarrow \quad m^2 = 1 \quad \Rightarrow \quad m = \pm 1 \\
& \quad m_1 = 1, \ m_2 = -1 \\
\text{general solution} & \quad y = c_1 e^x + c_2 e^{-x} \\
& \quad y(0) = c_1 + c_2 = 1 \\
& \quad y'(0) = c_1 - c_2 = 2 \\
\end{align*}
\]

\[
\begin{align*}
& \quad \frac{c_1 + c_2 = 1}{c_1 - c_2 = 2} \quad \Rightarrow \quad 2c_1 = 3 \\
& \quad c_1 = \frac{3}{2}, \quad c_2 = 1 - c_1 = 1 - \frac{3}{2} = -\frac{1}{2} \\
\end{align*}
\]

\[
y = \frac{3}{2} e^x - \frac{1}{2} e^{-x} \quad \text{is the solution to the IVP}
\]

(6) (10 pts) An RC-series circuit has a resistance of 25 ohms, capacitance of \(10^{-4}\) farads, and is attached to a battery that provides an oscillating input voltage of \((50 \cos(\omega t) + 50)\) volts. If the initial charge on the capacitor \(q(0) = 0\), write out the initial value problem governing the charge \(q(t)\) at time \(t\). Put the differential equation into standard form. **Do not solve the IVP.**

\[
R = 25, \quad C = 10^{-4}, \quad E = 50 \cos(\omega t) + 50
\]

\[
25 \frac{dq}{dt} + \frac{1}{10^{-4}} q = 50 \cos(\omega t) + 50 \]

\[
\frac{10^4 q}{25} = \frac{100}{25} \times 100 = 400
\]

In standard form, the IVP is

\[
\frac{dq}{dt} + 400q = 2 \cos(\omega t) + 2 \quad q(0) = 0
\]
(7) Consider the second order, linear, nonhomogeneous equation
\[ x^2 y'' + xy' - y = 2x. \]

(a) (5 pts) Verify that \( y_1 = \frac{1}{x} \) solves the associated homogeneous equation. (You may recall this from problem (2) on this exam.)

\[
\begin{align*}
y_1 &= \frac{1}{x} \\
y_1' &= \frac{1}{x^2} \\
y_1'' &= \frac{2}{x^3}
\end{align*}
\]

\[
x^2 \left( \frac{2}{x^3} \right) + x \left( \frac{1}{x^2} \right) - \frac{1}{x} = 0 \quad \text{so } y_1 \text{ solves}
\]

\[
x^2 \frac{2}{x^3} + x \frac{1}{x^2} - \frac{1}{x} = 0 \quad x^2 y_1'' + x y_1' - y_1 = 0
\]

(b) (5 pts) Verify that \( y_p = x \ln x \) solves the nonhomogeneous equation.

\[
\begin{align*}
y_p &= x \ln x \\
y_p' &= \ln x + 1 \\
y_p'' &= \frac{1}{x}
\end{align*}
\]

\[
x^2 \left( \frac{1}{x} \right) + x (\ln x + 1) - x \ln x = 2x \quad \text{so } y_p \text{ solves}
\]

\[
x^2 y_p'' + x y_p' - y_p = 2x
\]

(c) (5 pts) Jack and Diane are working together to find the general solution of the nonhomogeneous equation. Jack says that the solution looks like

\[ y = y_c + y_p \]

so his claim is that the general solution is

\[ y = \frac{c_1}{x} + x \ln x. \]

Diane disagrees. She thinks that \( y = \frac{c_1}{x} + x \ln x \) is NOT the general solution. Who is correct? And how do you know?

**Diane is correct. The equation is 2nd order. So a fundamental solution set for the associated homogeneous equation must have 2 lin. independent solutions in it. \( y_c \) is missing a \( y_2 \). \( y_c \) should look like \( c_1 y_1 + c_2 y_2 \).**