Exam 2 Math 2306 sec. 58

Spring 2016

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Determine whether each set of functions is linearly dependent or linearly independent on the indicated interval. Be sure that your conclusion is clearly stated and justified.

(a) $f_1(x) = x, \quad f_2(x) = 1, \quad f_3(x) = 2x-4 \quad I = (-\infty, \infty)$

Note that

$$2f_1(x) - 4f_2(x) - f_3(x) = 2x - 4 - (2x-4) = 0$$

for all x in I

We have a linear dependence relation w/ $c_1 = 2$, $c_2 = -4$, $c_3 = -1$ (not all zero).

Hence they are <u>linearly</u> <u>dependent</u>.

The wronskian would lead to the same conclusion (of course!).

(b) $y_1(x) = \sqrt{x}, \quad y_2(x) = \dfrac{1}{\sqrt{x}}, \quad I = (1, \infty)$

Take the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} \sqrt{x} & \frac{1}{\sqrt{x}} \\ \frac{1}{2\sqrt{x}} & \frac{-1}{2x^{3/2}} \end{vmatrix} = \frac{-\sqrt{x}}{2x^{3/2}} - \frac{1}{2\sqrt{x}\sqrt{x}}$$

$$= \frac{-1}{2x} - \frac{1}{2x} = \frac{-1}{x} \neq 0$$

Hence they are <u>linearly</u> <u>independent</u>,

(2) An RC series circuit has a 180 volt battery applied, a resistance of 3 ohms and a capacitance of $\frac{1}{60}$ farads. If the initial charge on the capacitor $q(0) = 0$, determine the charge on the capacitor $q(t)$ for $t > 0$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

Here $R = 3\,\Omega$

$C = \frac{1}{60}\,f$

$E = 180\,V$

$$3\frac{dq}{dt} + 60q = 180 \qquad q(0) = 0$$

In standard form $\quad \frac{dq}{dt} + 20q = 60$

$P(t) = 20$ so $\quad \mu = e^{\int P(t)dt} = e^{20t}$

$$\frac{d}{dt}\left[e^{20t}q\right] = 60e^{20t}$$

$$\Rightarrow \int \frac{d}{dt}\left[e^{20t}q\right]dt = \int 60e^{20t}\,dt$$

$$e^{20t}q = 3e^{20t} + k$$

$$q = 3 + ke^{-20t}$$

Since $q(0) = 0$, $\quad q(0) = 3 + k = 0 \Rightarrow k = -3$

The charge is $\quad q(t) = 3 - 3e^{-20t}.$

(3) Determine the general solution of the second order equation for which one solution is given.

$x^2y''+2xy'-12y=0, \quad y_1(x)=x^3$

Standard form

$$y''+\frac{2}{x}y'-\frac{12}{x^2}y=0$$

$$P(x)=\frac{2}{x} \Rightarrow -\int P(x)dx = -\int \frac{2}{x}dx = -2\ln x = \ln x^{-2}.$$

$y_2 = uy_1$ where $u=\int \frac{e^{-\int P(x)dx}}{y_1^2}dx = \int \frac{x^{-2}}{(x^3)^2}dx = \int x^{-8}dx$

$u=\frac{x^{-7}}{-7}$. We can take $y_2 = x^{-7}\cdot x^3 = x^{-4}$

The general solution is $y=C_1x^3+C_2x^{-4}$.

(4) Find the general solution of the first order linear equation.

$\frac{dy}{dx}+2xy=4x$ $\quad P(x)=2x$, $\mu = e^{\int P(x)dx} = e^{\int 2x\,dx} = e^{x^2}$

$\frac{d}{dx}\left[e^{x^2}y\right] = 4xe^{x^2}$ $\quad$ If $u=x^2$, then $du=2x\,dx$

so $\int 4xe^{x^2}dx = 2\int e^u du = 2e^u + C$

$$\int \frac{d}{dx}\left[e^{x^2}y\right]dx = \int 4xe^{x^2}dx$$

$$e^{x^2}y = 2e^{x^2}+C \Rightarrow y = 2+Ce^{-x^2}$$

(5) (a) Show that $y_1 = e^x$, $y_2 = e^{2x}$ is a fundamental solution set for the ODE $y'' - 3y' + 2y = 0$.

$y_1 = e^x$ $\quad y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x = 0$ $\quad$ both are solutions.

$y_1' = e^x$

$y_1'' = e^x$ $\quad y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$

$y_2 = e^{2x}$ $\quad W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$

$y_2' = 2e^{2x}$

$y_2'' = 4e^{2x}$

We have 2 linearly independent solutions, hence they are a fund. soln. set.

(b) Verify that $y_p = 2x + 3$ solves the nonhomogeneous equation $y'' - 3y' + 2y = 4x$.

$y_p' = 2$ $\quad y_p'' - 3y_p' + 2y_p = 0 - 3(2) + 2(2x+3)$

$y_p'' = 0$ $\quad = -6 + 4x + 6$

$= 4x$ as required

Hence $y_p = 2x+3$ solves the nonhomogeneous ODE.

(c) Find the solution of the IVP $y'' - 3y' + 2y = 4x$, with $y(0) = 4$, $y'(0) = 2$.

$y = c_1 e^x + c_2 e^{2x} + 2x + 3$ (General solution).

$y' = c_1 e^x + 2c_2 e^{2x} + 2$

$y(0) = c_1 + c_2 + 3 = 4 \Rightarrow c_1 + c_2 = 1$

$y'(0) = c_1 + 2c_2 + 2 = 2 \quad\quad c_1 + 2c_2 = 0$ subtract

$-c_2 = 1$

$c_1 - 1 = 1 \Rightarrow c_1 = 2$

The solution to the IVP is

$$y = 2e^x - e^{2x} + 2x + 3.$$

(6) A tank initially contains 200 gallons of water into which 5 pounds of salt is dissolved. Brine containing 1 lb/gal of salt is pumped in at a rate of 3 gal/min. The solution is well mixed and is pumped out at the same rate. Determine the amount of salt $A(t)$ in lbs at time t in minutes for all $t > 0$.

$$\frac{dA}{dt} = r_i c_i - r_o c_o \quad \text{where} \quad c_o = \frac{A}{V(0) + (r_i - r_o)t}$$

Here $r_i = 3 \frac{\text{gal}}{\text{min}}$, $c_i = 1 \frac{\text{lb}}{\text{gal}}$, $r_o = r_i = 3 \frac{\text{gal}}{\text{min}}$

so $c_o = \frac{A}{200 + 0} = \frac{A}{200}$

$$\frac{dA}{dt} = 3 \cdot 1 - 3 \cdot \frac{A}{200}, \quad A(0) = 5$$

$$A' + \frac{3}{200} A = 3 \qquad P(t) = \frac{3}{200} \qquad \mu = e^{\int \frac{3}{200} dt} = e^{\frac{3}{200}t}$$

$$\frac{d}{dt}\left[e^{\frac{3}{200}t} A \right] = 3 e^{\frac{3}{200}t}$$

$$e^{\frac{3}{200}t} A = \int 3 e^{\frac{3}{200}t} \, dt = 200 e^{\frac{3}{200}t} + C$$

$$A = 200 + C e^{-\frac{3}{200}t}$$

From $A(0) = 5$, $A(0) = 200 + C = 5 \Rightarrow C = -195$

Finally, $A(t) = 200 - 195 e^{\frac{-3}{200}t}$.