Exam 2 Math 2306 sec. 59

Spring 2016

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5” $\times$ 11”) of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) A tank initially contains 200 gallons of water into which 5 pounds of salt is dissolved. Brine containing 1 lb/gal of salt is pumped in at a rate of 3 gal/min. The solution is well mixed and is pumped out at the same rate. Determine the amount of salt $A(t)$ in lbs at time t in minutes for all $t > 0$.

$\frac{dA}{dt} = r_i c_i - r_o c_o$ where $c_o = \frac{A}{V(0) + (r_i - r_o)t}$

Here $r_i = 3 \frac{gal}{min}$, $c_i = 1 \frac{lb}{gal}$ and $r_o = r_i = 3 \frac{gal}{min}$

So $c_o = \frac{A}{200 + 0} = \frac{A}{200}$ also, $A(0) = 5$ lb

$$\frac{dA}{dt} = 3 \cdot 1 - 3 \cdot \frac{A}{200} \Rightarrow \frac{dA}{dt} + \frac{3}{200} A = 3, \quad A(0) = 5$$

$$P(t) = \frac{3}{200} \Rightarrow \mu = e^{\int P(t)\,dt} = e^{\frac{3}{200}t}$$

$$\frac{d}{dt}\left[e^{\frac{3}{200}t} A\right] = 3e^{\frac{3}{200}t}$$

$$e^{\frac{3}{200}t} A = \int 3e^{\frac{3}{200}t}\,dt = 200\, e^{\frac{3}{200}t} + C$$

$$A = 200 + C\, e^{-\frac{3}{200}t}$$

From $A(0) = 5$, $A(0) = 200 + C = 5 \Rightarrow C = -195$

The amount of salt @ time t is

$$A(t) = 200 - 195\, e^{-\frac{3}{200}t} \text{ pounds.}$$

(2) Determine the general solution of the second order equation for which one solution is given.

Standard form

$x^2y''+2xy'-12y=0, \quad y_1(x)=x^{-4}$

$$y''+\frac{2}{x}y'-\frac{12}{x^2}y=0 \qquad P(x)=\frac{2}{x}$$

$y_2=uy_1$ where

$$u=\int\frac{e^{-\int P(x)dx}}{y_1^2}dx=\int\frac{e^{-\int\frac{2}{x}dx}}{(x^{-4})^2}dx=\int\frac{e^{-2\ln x}}{x^{-8}}dx$$

$$=\int\frac{x^{-2}}{x^{-8}}dx=\int x^6\,dx=\frac{1}{7}x^7$$

We can ignore the factor $\frac{1}{7}$ and take

$$y_2=x^7\cdot x^{-4}=x^3$$

The general solution is $\underline{\underline{y=c_1x^{-4}+c_2x^3}}$

(3) Find the general solution of the first order linear equation.

$$\frac{dy}{dx}+3x^2y=6x^2 \qquad P(x)=3x^2 \Rightarrow \mu=e^{\int P(x)dx}=e^{x^3}$$

$$\frac{d}{dx}\left[e^{x^3}y\right]=6x^2e^{x^3}$$

If $u=x^3$, $du=3x^2dx$

$$\int 6x^2e^{x^3}dx=2\int e^u\,du=2e^u+C$$

$$e^{x^3}y=\int 6x^2e^{x^3}dx=2e^{x^3}+C$$

So $\underline{\underline{y=2+Ce^{-x^3}}}$.

(4) Determine whether each set of functions is linearly dependent or linearly independent on the indicated interval. Be sure that your conclusion is clearly stated and justified.

(a) $y_1(x) = e^x$, $y_2(x) = x^2e^x$ $I = (-\infty, \infty)$

Computing the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & x^2e^x \\ e^x & 2xe^x + x^2e^x \end{vmatrix}$$

$$= 2xe^{2x} + x^2e^{2x} - x^2e^{2x} = 2xe^{2x} \neq 0$$

The Wronskian is not identically zero.

Hence the functions are <u>linearly independent</u>.

(b) $f_1(x) = x+1$, $f_2(x) = x-1$, $f_3(x) = 4x$, $I = (1, \infty)$

Note that

$$2f_1(x) + 2f_2(x) - f_3(x) =$$

$$2(x+1) + 2(x-1) - 4x = 4x - 4x = 0$$

for all x in I.

We have a linear dependence relation with $c_1 = 2$, $c_2 = 2$, $c_3 = -1$ (not all zero).

Hence the functions are <u>linearly dependent</u>.

(5) (a) Show that $y_1 = e^x$, $y_2 = e^{3x}$ is a fundamental solution set for the ODE $y'' - 4y' + 3y = 0$.

$y_1 = e^x$, $y_1' = e^x$ $y_1'' = e^x$

$y_1'' - 4y_1' + 3y_1 = e^x - 4e^x + 3e^x = 0$

$y_2 = e^{3x}$, $y_2' = 3e^{3x}$, $y_2'' = 9e^{3x}$

$y_2'' - 4y_2' + 3y_2 =$

$9e^{3x} - 12e^{3x} + 3e^{3x} = 0$

$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix}$

$= 3e^{4x} - e^{4x}$

$= 2e^{4x} \neq 0$

We have 2 linearly independent solutions and hence a fundamental solution set.

(b) Verify that $y_p = 3x + 4$ solves the nonhomogeneous equation $y'' - 4y' + 3y = 9x$.

$y_p' = 3$

$y_p'' = 0$

$y_p'' - 4y_p' + 3y_p$

$= 0 - 4 \cdot 3 + 3(3x+4)$

$= -12 + 9x + 12 \quad = 9x$

So y_p solves the nonhomogeneous ODE.

(c) Find the solution of the IVP $y'' - 4y' + 3y = 9x$, with $y(0) = 4$, $y'(0) = 1$.

$y = c_1 e^x + c_2 e^{3x} + 3x + 4$ general solution

$y' = c_1 e^x + 3c_2 e^{3x} + 3$

$y(0) = c_1 + c_2 + 4 = 4$

$y'(0) = c_1 + 3c_2 + 3 = 1$

$\Rightarrow$

$c_1 + c_2 = 0$

$c_1 + 3c_2 = -2$

$-2c_2 = 2$ $\quad c_2 = -1$

$c_1 - 1 = 0 \Rightarrow c_1 = 1$

The solution to the IVP is

$$\boxed{y = e^x - e^{3x} + 3x + 4.}$$

(6) An RC series circuit has a 240 volt battery applied, a resistance of 5 ohms and a capacitance of $\frac{1}{60}$ farads. If the initial charge on the capacitor $q(0) = 0$, determine the charge on the capacitor $q(t)$ for $t > 0$.

$$R \frac{dq}{dt} + \frac{1}{C} q = E$$

Here $R = 5\,\Omega$

$C = \frac{1}{60}$ f

$E = 240$ V

$$5 \frac{dq}{dt} + 60q = 240$$

$$\frac{dq}{dt} + 12q = 48 \qquad q(0) = 0$$

$P(t) = 12$ so $\mu = e^{\int P(t)dt} = e^{12t}$

$$\frac{d}{dt}\left[e^{12t} q\right] = 48 e^{12t}$$

$$e^{12t} q = \int 48 e^{12t} dt = 4e^{12t} + k$$

$$q = 4 + k e^{-12t}$$

As $q(0) = 0$, $\quad q(0) = 4 + k = 0 \Rightarrow k = -4$

The charge on the capacitor is

$$\underline{\underline{q(t) = 4 - 4e^{-12t}}}.$$