Exam 2 Math 2306 sec. 60

Fall 2018

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

1. Let a be a constant, and consider the pair of functions $y_1 = x^2 + ax$ and $y_2 = x^2$ defined in the interval $(0, \infty)$.

(a) Find the Wronskian of y_1 and y_2.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 + ax & x^2 \\ 2x + a & 2x \end{vmatrix} = 2x^3 + 2ax^2 - (2x^3 + ax^2)$$
$$= ax^2$$

$$W = ax^2$$

(b) For what values of a is the pair of functions linearly independent? (Justify)

They are linearly independent if $W \neq 0$.
This holds for all $a \neq 0$

2. Find the general solution of the second order, linear differential equation. One solution is provided. Assume $x > 0$.

$$x^2y'' - 5xy' + 9y = 0, \qquad y_1 = x^3$$

Standard form

$y'' - \frac{5}{x}y' + \frac{9}{x^2}y \qquad P(x) = \frac{-5}{x}$

$y_2 = uy_1$

where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}$

$e^{-\int P(x)dx} = e^{-\int \frac{-5}{x}dx} = e^{\int \frac{5}{x}dx} = e^{5\ln x} = x^5$

$(y_1)^2 = (x^3)^2 = x^6$

$u = \int \frac{x^5}{x^6}\,dx = \int \frac{1}{x}\,dx = \ln x$

so $y_2 = x^3 \ln x$

and the general solution is

$$y = C_1 x^3 + C_2 x^3 \ln x$$

3. A tank initially contains 100 liters of water into which 5 kg of salt is dissolved. Salt water containing 0.1 kg/L of salt flows in at a rate of 10 L/min, and the well mixed solution is pumped out at the same rate. Determine the amount of salt $A(t)$ in kg in the tank at the time t in minutes.

$V(0) = 100\,L$ and $A(0) = 5$ kg

$r_i = 10 \frac{L}{min}$ and $r_o = 10$ L min

so $V(t) = 100\,L$ for all t

$c_i = 0.1 \frac{kg}{L} \Rightarrow r_i c_i = 1 \frac{kg}{min}$

$$\frac{dA}{dt} + r_o c_o = r_i c_i \qquad \frac{dA}{dt} + 10\frac{A}{100} = 1$$

$$p(t) = \frac{1}{10} \qquad \mu = e^{\int \frac{1}{10} dt} = e^{\frac{1}{10}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{10}t} A\right) = e^{\frac{1}{10}t} \Rightarrow e^{\frac{1}{10}t} A = 10 e^{\frac{1}{10}t} + C$$

$$A = 10 + C e^{-\frac{1}{10}t}$$

$$A(0) = 5 = 10 + C \Rightarrow C = -5$$

The amount of salt is

$$\boxed{A(t) = 10 - 5 e^{-\frac{1}{10}t} \text{ kg}}$$

4. Consider the nonhomogeneous, linear differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 8x + 1, \quad \text{for} \quad 0 < x < \infty$$

(a) Verify that $y_{p_1} = 2x^2$ is a solution of $\quad x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 8x.$

$y_{p_1}' = 4x$

$y_{p_1}'' = 4$

$x(4) + (4x) \stackrel{?}{=} 8x$

$8x = 8x$

true

y_{p_1} is a solution

(b) Verify that $y_{p_2} = x$ is a solution of $\quad x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 1.$

$y_{p_2}' = 1$

$y_{p_2}'' = 0$

$x(0) + (1) \stackrel{?}{=} 1$

$1 = 1$

true

y_{p_2} is a solution

(c) Given that the general solution of $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$ is $y = c_1 + c_2 \ln x$, find the solution of the initial value problem

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 8x + 1, \quad y(1) = 1, \quad y'(1) = 0$$

$y = y_c + y_p$ and $y_p = y_{p_1} + y_{p_2}$

$y = c_1 + c_2 \ln x + 2x^2 + x$

$y' = \frac{c_2}{x} + 4x + 1$

$y(1) = c_1 + c_2 \ln 1 + 2 \cdot 1^2 + 1 = 1$

$c_1 + 3 = 1$

$c_1 = -2$

$y'(1) = \frac{c_2}{1} + 4 \cdot 1 + 1 = 0$

$c_2 = -5$

The solution to the IVP is

$$\boxed{y = -2 - 5\ln x + 2x^2 + x}$$

5. Solve the initial value problem

$$y\,dx + \left(2x - \frac{e^y}{y}\right) dy = 0 \qquad y(1) = 1$$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2 \qquad$ not exact

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-1}{2x - e^y/y}$ depends on y

$\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1}{y}$ depends only on $y \qquad \mu = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

$y^2 dx + (2xy - e^y) dy = 0$

Solutions satisfy $F(x,y) = C$ where

$F(x,y) = \int y^2 dx = \int (2xy - e^y) dy$

$F(x,y) = y^2 x + g(y) \qquad \frac{\partial F}{\partial y} = 2yx + g'(y) = 2xy - e^y$

$g'(y) = -e^y$

$g(y) = -e^y$

Solutions are $x^2y - e^y = C$

Apply $y(1) = 1 \qquad 1^2 \cdot 1 - e^1 = C \implies C = 1 - e$

$$\boxed{\text{The solution to the IVP is defined by } \quad x^2y - e^y = 1 - e}$$