Exam 2 Math 3260 sec. 51

Spring 2020

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 20 points each. Do any 5 problems (I'll count your best 5). **No calculator use is allowed, and no calculator use is needed. Use of a textbook, notes, calculator or smart device is strictly prohibited. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

1. Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$. A and the rref of A are given.

$$A = \begin{bmatrix} -2 & 2 & -3 & -2 & -8 \\ 3 & -3 & 3 & 1 & 10 \\ 2 & -2 & 2 & 0 & 4 \end{bmatrix} \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

If $A\vec{x}=\vec{0}$, $x_1 = x_2 - 6x_5$

$x_3 = 4x_5$

$x_4 = -4x_5$

x_2, x_5 - free

$$\vec{x} = x_2\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} + x_5\begin{bmatrix}-6\\0\\4\\-4\\1\end{bmatrix}$$

A Basis for:

Nul A is $\left\{\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}6\\0\\4\\-4\\1\end{bmatrix}\right\}$

Col A is $\left\{\begin{bmatrix}-2\\3\\2\end{bmatrix}, \begin{bmatrix}-3\\3\\2\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}$

Pivot columns are 1, 3, 4

View vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix (called the scalar product) that is usually written as a number without brackets. The matrix product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix (called the outer product).

2. Consider the following vectors.

$$\mathbf{u} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad (a, b, c \text{ are real numbers}).$$

(a) Compute $\mathbf{u}^T\mathbf{v}$.

$$\vec{u}^T \vec{v} = \begin{bmatrix} 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a - 2b + 3c$$

$1 \times 3 \quad 3 \times 1$

1×1

(b) Compute $\mathbf{u}\mathbf{v}^T$.

$$\vec{u}\vec{v}^T = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -2a & -2b & -2c \\ 3a & 3b & 3c \end{bmatrix}$$

$3 \times 1 \quad 1 \times 3$

3×3

(c) Compute the 2×2 matrix $\mathbf{x}\mathbf{x}^T$.

$$\vec{x}\vec{x}^T = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -6 & 4 \end{bmatrix}$$

$2 \times 1 \quad 1 \times 2$

2×2

(d) Determine whether $\mathbf{x}\mathbf{x}^T$ is singular or nonsingular. If nonsingular, compute its inverse.

$$\det(\vec{x}\vec{x}^T) = 9(4) - (-6)^2 = 36 - 36 = 0$$

$\vec{x}\vec{x}^T$ is singular

3. Answer each short computational problem. Here, I_n is the $n \times n$ identity matrix.

(a) Suppose A, B, and C are $n \times n$ invertible matrices. Does the equation

$$C^{-1}(X + A)B^{-1} = I_n$$

have a solution X? If so, find it.

$C^{-1}(X+A)B^{-1} = I \Rightarrow CC^{-1}(X+A)B^{-1}B = CIB$

$X + A = CB \Rightarrow \boxed{X = CB - A}$

(b) Suppose A and B are 3×3 matrices with $\det(A) = -2$ and $\det(B) = 5$. Evaluate each of

(i) $\det(A^3) =$ $(-2)^3 = -8$

(ii) $\det(B^T A) =$ $5(-2) = -10$

(iii) $\det(A^{-1}B) =$ $\frac{-1}{2}(5) = \frac{-5}{2}$

(iv) $\det(B^{-1}AB) =$ $\frac{1}{5}(-2)5 = -2$

(c) Suppose $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 4$. Evaluate each determinant.

$\det \begin{bmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{bmatrix} =$ 12

$3R_3 \to R_3$

$\det \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix} =$ -4

$R_1 \leftrightarrow R_3$

$\det \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix} =$ 4

$2R_1 + R_2 \to R_2$

4. Determine the values of the parameter s for which the system of equations has a unique solution. For those values of s use Crammer's rule to obtain the solutions X and Y.

$$\begin{array}{rcrcr} sX & - & 2sY & = & 1 \\ 3X & + & 12sY & = & -1 \end{array} \qquad \underbrace{\begin{bmatrix} s & -2s \\ 3 & 12s \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 12s^2 + 6s = 6s(2s+1)$ $\quad \det(A) = 0$ if $s = 0$ or $s = \frac{-1}{2}$.

There is a unique solution if $s \neq 0$ and $s \neq \frac{-1}{2}$.

$$\det(A_1(\vec{b})) = \begin{vmatrix} 1 & -2s \\ -1 & 12s \end{vmatrix} = 12s - 2s = 10s$$

$$\det(A_2(\vec{b})) = \begin{vmatrix} s & 1 \\ 3 & -1 \end{vmatrix} = -s - 3 = -(s+3)$$

For $s \neq 0$ and $s \neq \frac{-1}{2}$ $\qquad X = \dfrac{10s}{6s(2s+1)} = \dfrac{5}{3(2s+1)}$

$$Y = \frac{-(s+3)}{6s(2s+1)}$$

5. Show that the linear transformation is invertible, and find a formula for T^{-1}. $\quad T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x_1, x_2) = (5x_1 + 4x_2, 2x_1 + 2x_2)$.

Standard matrix $A = [T(\vec{e}_1)\ T(\vec{e}_2)] = \begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}$

$\det(A) = 10 - 8 = 2 \neq 0$ $\qquad T^{-1}$ exists.

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & \frac{5}{2} \end{bmatrix}$$

$$T^{-1}(\vec{x}) = A^{-1}\vec{x} = \begin{bmatrix} 1 & -2 \\ -1 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + \frac{5}{2}x_2 \end{bmatrix}$$

$$T^{-1}(x_1, x_2) = \left(x_1 - 2x_2,\ -x_1 + \frac{5}{2}x_2\right)$$

6. Find a basis for each subspace of the indicated vector space.

(a) The set of vectors in $\mathbb{R}^3$ that are on the plane $x + 2y - 5z = 0$. (Hint: Think of the equation as a homogeneous linear system.)

$x = -2y + 5z$, y, z - free

$$\vec{x} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

A basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) The set of matrices in $M^{2\times2}$ of the form $\begin{bmatrix} a & b \\ 0 & 2b \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ 0 & 2b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

A basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}$

7. Recall that $\mathbb{P}_n$ denotes the vectors space consisting of all polynomials of degree at most n.

Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ defined by $T(\mathbf{p}) = \begin{bmatrix} 2\mathbf{p}(0) \\ -\mathbf{p}(0) \end{bmatrix}$.

(a) Evaluate $T(\mathbf{p}_1)$ if $\mathbf{p}_1(t) = 2 + t - t^2$

$\vec{p}_1(0) = 2$ $\quad T(\vec{p}_1) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

(b) Evaluate $T(\mathbf{p}_2)$ if $\mathbf{p}_2(t) = 2t$

$\vec{p}_2(0) = 0$ $\quad T(\vec{p}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) Evaluate $T(\mathbf{p}_3)$ if $\mathbf{p}_3(t) = 4t^2 + 2t - 3$

$\vec{p}_3(0) = -3$ $\quad T(\vec{p}_3) = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

(d) Find a vector $\mathbf{u}$ in $\mathbb{R}^2$ that spans the range of T.

$T(\vec{p}) = \vec{p}(0) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ we can take $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(e) Find any vector $\mathbf{p}$ in $\mathbb{P}_2$ that is in the kernel of T

Using part (b) above $T(\vec{p}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So we can choose $\vec{p} = \vec{p}_2 = 2t$