Exam 2 Math 3260 sec. 55

Spring 2020

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 20 points each. Do any 5 problems (I'll count your best 5). **No calculator use is allowed, and no calculator use is needed. Use of a textbook, notes, calculator or smart device is strictly prohibited. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

1. Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$. A and the rref of A are given.

$$A = \begin{bmatrix} 3 & -6 & 1 & 6 & 14 \\ -1 & 2 & 0 & -1 & -4 \\ 2 & -4 & 1 & 5 & 10 \end{bmatrix} \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns are 1 and 3

If $A\vec{x} = \vec{0}$

$x_1 = 2x_2 - x_4 - 4x_5$

$x_3 = -3x_4 - 2x_5$

x_2, x_4, x_5 - free

$$\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 6 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

A basis for Nul A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Col A is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

View vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix (called the scalar product) that is usually written as a number without brackets. The matrix product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix (called the outer product).

2. Consider the following vectors.

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad (a, b, c \text{ are real numbers}).$$

(a) Compute $\mathbf{u}^T\mathbf{v}$.

$$\vec{u}^T \vec{v} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} = -2a + 4b + 3c$$

1×3 3×1

1×1

(b) Compute $\mathbf{u}\mathbf{v}^T$.

$$\vec{u}\vec{v}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -2a & 4a & 3a \\ -2b & 4b & 3b \\ -2c & 4c & 3c \end{bmatrix}$$

3×1 1×3

3×3

(c) Compute the 2×2 matrix $\mathbf{x}\mathbf{x}^T$.

$$\vec{x}\vec{x}^T = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} 5 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 10 \\ 10 & 4 \end{bmatrix}$$

(d) Determine whether $\mathbf{x}\mathbf{x}^T$ is singular or nonsingular. If nonsingular, compute its inverse.

$$\det(\vec{x}\vec{x}^T) = 100 - 100 = 0$$

$\vec{x}\vec{x}^T$ is singular

3. Answer each short computational problem. Here, I_n is the $n \times n$ identity matrix.

(a) Suppose A, B, and C are $n \times n$ invertible matrices. Does the equation

$$B^{-1}(X + C)A^{-1} = I_n$$

have a solution X? If so, find it.

$B^{-1}(X+C)A^{-1} = I \Rightarrow BB^{-1}(X+C)A^{-1}A = BIA$

$X + C = BA$

$\Rightarrow \boxed{X = BA - C}$

(b) Suppose A and B are 3×3 matrices with $\det(A) = 4$ and $\det(B) = -3$. Evaluate each of

(i) $\det(B^3) =$ $(-3)^3 = -27$

(ii) $\det(A^T B) =$ $4(-3) = -12$

(iii) $\det(B^{-1}A) =$ $-\frac{1}{3}(4) = \frac{-4}{3}$

(iv) $\det(A^{-1}BA) =$ $\frac{1}{4}(-3)4 = -3$

(c) Suppose $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -5$. Evaluate each determinant.

$\det \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix} =$ -10

$2R_2 \to R_2$

$\det \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix} =$ 5

$R_1 \leftrightarrow R_3$

$\det \begin{bmatrix} a & b & c \\ d-a & e-b & f-c \\ g & h & i \end{bmatrix} =$ -5

$-R_1 + R_2 \to R_2$

4. Determine the values of the parameter s for which the system of equations has a unique solution. For those values of s use Crammer's rule to obtain the solutions X and Y.

$$\begin{array}{rcrcl} 2sX & + & 3Y & = & 2 \\ 6X & + & sY & = & 1 \end{array}$$

$$\underbrace{\begin{bmatrix} 2s & 3 \\ 6 & s \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 2s^2 - 18 = 2(s^2 - 9)$ $\qquad \det(A) = 0 \Rightarrow s = \pm 3$.

There is a unique solution if $s \neq \pm 3$

$$\det(A_1(\vec{b})) = \begin{vmatrix} 2 & 3 \\ 1 & s \end{vmatrix} = 2s - 3$$

$$\det(A_2(\vec{b})) = \begin{vmatrix} 2s & 2 \\ 6 & 1 \end{vmatrix} = 2s - 12 = 2(s-6)$$

For $s \neq \pm 3$ $\qquad X = \dfrac{2s-3}{2(s^2-9)}$

$$y = \frac{2(s-6)}{2(s^2-9)} = \frac{s-6}{s^2-9}$$

5. Show that the linear transformation is invertible, and find a formula for T^{-1}. $\quad T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x_1, x_2) = (3x_1 + 5x_2, 4x_1 + 6x_2)$.

Standard matrix $A = [T(\vec{e}_1) \; T(\vec{e}_2)] = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$\det(A) = 18 - 20 = -2 \neq 0$ $\qquad T^{-1}$ exists

$$A^{-1} = \frac{-1}{2}\begin{bmatrix} 6 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix}$$

$$T^{-1}(\vec{x}) = A^{-1}\vec{x} = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + \frac{5}{2}x_2 \\ 2x_1 - \frac{3}{2}x_2 \end{bmatrix}$$

$$T^{-1}(x_1, x_2) = \left(-3x_1 + \frac{5}{2}x_2,\; 2x_1 - \frac{3}{2}x_2\right)$$

6. Find a basis for each subspace of the indicated vector space.

(a) The set of matrices in $M^{2\times 2}$ of the form $\begin{bmatrix} 2a & 0 \\ a & b \end{bmatrix}$.

$$\begin{bmatrix} 2a & 0 \\ a & b \end{bmatrix} = a\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

a basis is $\left\{ \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(b) The set of vectors in $\mathbb{R}^3$ that are on the plane $x - 3y + 4z = 0$. (Hint: Think of the equation as a homogeneous linear system.)

$x = 3y - 4z$
y, z - free

$$\vec{x} = y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

a basis is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

7. Recall that $\mathbb{P}_n$ denotes the vectors space consisting of all polynomials of degree at most n.

Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ defined by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ -3\mathbf{p}(0) \end{bmatrix}$.

(a) Evaluate $T(\mathbf{p}_1)$ if $\mathbf{p}_1(t) = 2 + t - t^2$

$\vec{p}_1(0) = 2$ $\qquad$ $T(\vec{p}_1) = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

(b) Evaluate $T(\mathbf{p}_2)$ if $\mathbf{p}_2(t) = 2t$

$\vec{p}_2(0) = 0$ $\qquad$ $T(\vec{p}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) Evaluate $T(\mathbf{p}_3)$ if $\mathbf{p}_3(t) = 4t^2 + 2t - 3$

$\vec{p}_3(0) = -3$ $\qquad$ $T(\vec{p}_3) = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

(d) Find a vector $\mathbf{u}$ in $\mathbb{R}^2$ that spans the range of T.

$T(\vec{p}) = \vec{p}(0) \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ one choice is $\vec{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

(e) Find any vector $\mathbf{p}$ in $\mathbb{P}_2$ that is in the kernel of T.

Using $\vec{p}_2$ above, $T(\vec{p}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

so one choice is $\vec{p} = \vec{p}_2 = 2t$