Exam 3 Math 2254H sec. 015H

Spring 2015

Name: 4 points Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature:

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. No use of notes, books, or calculator is allowed. Illicit use of notes, a book, calculator, or any smart device will result in a grade of zero on this exam and may result in a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Determine if the geometric series is convergent or divergent. If convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n-1}}{4^{n+2}}$

$2^{2n-1} = \frac{1}{2}(2^2)^n = \frac{1}{2} \cdot 4^n$

$4^{n+2} = 4^2 \cdot 4^n = 16 \cdot 4^n$

$= \displaystyle\sum_{n=1}^{\infty} \frac{1}{32}(1^n)$

$r = 1 \Rightarrow |r| \geq 1$

This series is divergent.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^{n+2}}{2^{3n-1}}$

$(-3)^{n+2} = (-3)^2 (-3)^n = 9 \cdot (-3)^n$

$2^{3n-1} = \frac{1}{2}(2^3)^n = \frac{1}{2} \cdot 8^n$

$= \displaystyle\sum_{n=0}^{\infty} 18 \left(\frac{-3}{8}\right)^n$

$r = \frac{-3}{8} \Rightarrow |r| < 1$, $a = 18$

(convergent)

$= \dfrac{18}{1 - \left(\frac{-3}{8}\right)} = \dfrac{8 \cdot 18}{8 + 3} = \dfrac{144}{11}$

(2) Find a set of parametric equations $x = f(t)$, $y = g(t)$ with the condition $0 \le t \le 1$ that defines the straight line path beginning at the point $(2, 4)$ and ending at the point $(4, -1)$.

Let $x = at + b$

$y = ct + d$

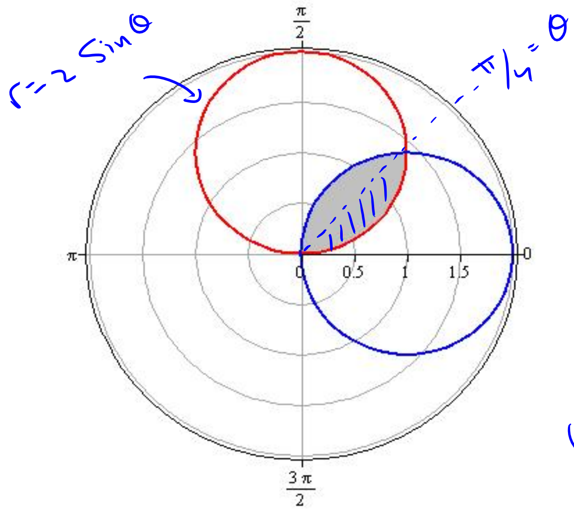
$x(0) = b = 2$ and $y(0) = d = 4$

$x(1) = a + b = 4 \Rightarrow a = 4 - b = 2$

$y(1) = c + d = -1 \Rightarrow c = -1 - d = -5$

$$\boxed{\begin{aligned} x(t) &= 2t + 2 \\ y(t) &= -5t + 4 \end{aligned} \quad , \quad 0 \le t \le 1}$$

(3) Find the area of the region common to the two polar curves $r = 2\sin\theta$ and $r = 2\cos\theta$.



$2\sin\theta = 2\cos\theta \Rightarrow$

$\tan\theta = 1$

The first positive solution is $\theta = \frac{\pi}{4}$

Using the symmetry, we can find the area of the bottom half of the region and double it.

$$\frac{1}{2}A = \frac{1}{2}\int_0^{\pi/4} (2\sin\theta)^2\, d\theta$$

$$\Rightarrow A = \int_0^{\pi/4} 4\sin^2\theta\, d\theta$$

$$= 2\int_0^{\pi/4} (1-\cos 2\theta)\, d\theta$$

$$= 2\theta - \sin 2\theta \Big|_0^{\pi/4}$$

$$= 2\cdot\frac{\pi}{4} - \sin\frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} - 1$$

(4) Determine if each sequence is convergent or divergent. If convergent, find its limit.

(a) $a_n = \dfrac{(-1)^n}{2\sqrt{n}}$

note $|a_n| = \left|\dfrac{(-1)^n}{2\sqrt{n}}\right| = \dfrac{1}{2\sqrt{n}}$

so $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \dfrac{1}{2\sqrt{n}} = 0$

Hence $\lim\limits_{n\to\infty} a_n = 0$ (squeeze thm)

(b) $b_n = \ln(2n+1) - \ln(n+2)$ $= \ln\left(\dfrac{2n+1}{n+2}\right)$

Note $\lim\limits_{n\to\infty} \dfrac{2n+1}{n+2} = \lim\limits_{n\to\infty} \dfrac{2n+1}{n+2} \cdot \dfrac{\frac{1}{n}}{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \dfrac{2+\frac{1}{n}}{1+\frac{2}{n}} = 2$

As $\ln x$ is continuous @ $x=2$

$\lim\limits_{n\to\infty} b_n = \ln 2$

(c) $c_1 = 1, \quad c_{n+1} = 4 - c_n, \quad$ for $\quad n \geq 1$

$c_2 = 4 - 1 = 3$

$c_3 = 4 - 3 = 1$

$c_4 = 4 - 1 = 3$

$\vdots$

This sequence is $\{1, 3, 1, 3, 1, 3, \ldots\}$.

It diverges - it oscillates.

(5) Use the integral test to determine if the series is convergent or divergent.

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

Let $f(x) = x^2 e^{-x^3}$. Clearly f is positive and continuous.

$$f'(x) = 2x e^{-x^3} - 3x^4 e^{-x^3} = x e^{-x^2}(2-3x^3)$$

for $x \geq 1$ $2-3x^3 < 0$ so f is decreasing.

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3}\,dx$$

$$= \lim_{t\to\infty} \frac{-1}{3} e^{-x^3} \Big|_1^t$$

$$= \lim_{t\to\infty} \frac{-1}{3} e^{-t^3} + \frac{1}{3} e^{-1^3} = 0 + \frac{1}{3}e^{-1}$$

The integral converges. So the series also converges.

Note $\int x^2 e^{-x^3}\,dx = -\frac{1}{3}\int e^u\,du$ $\quad u = -x^3$

$du = -3x^2\,dx$

$= -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-x^3} + C$ $\quad -\frac{1}{3}\,du = x^2\,dx$

(6) Determine all values of x that satisfy the equation.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n} = \frac{3}{2}$$

If the series converges, it converges to $\dfrac{1}{1-\left(\frac{x-1}{3}\right)}$ since it's geometric w/ $a=1$, $r=\dfrac{x-1}{3}$

$$\frac{1}{1-\frac{x-1}{3}} = \frac{3}{2} \Rightarrow \frac{3}{3-(x-1)} = \frac{3}{2}$$

$$\Rightarrow \frac{3}{4-x} = \frac{3}{2}$$

$$\Rightarrow 4-x=2 \Rightarrow x=2.$$

There is one solution, $x=2$.

* Note $\left|\frac{2-1}{3}\right| = \left|\frac{1}{3}\right| < 1$ so convergence is guaranteed when $x=2$.