Exam 3 Math 2254 sec. 001

Summer 2015

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

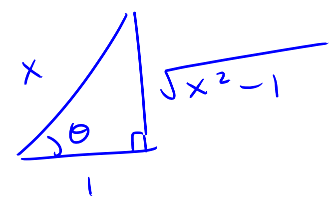
Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

INSTRUCTIONS: There are 10 problems worth 10 points each. No use of notes, books, or calculator is allowed. Illicit use of notes, a book, calculator, or any smart device will result in a grade of zero on this exam and may result in a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Evaluate the indefinite integral.

$$\int \frac{dx}{x^2\sqrt{x^2-1}}$$

Trig. Substitution



$x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$\sqrt{x^2-1} = \tan\theta$

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta \tan\theta \, d\theta}{(\sec\theta)^2 \tan\theta}$$

$$= \int \frac{1}{\sec\theta} \, d\theta$$

$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + C$$

$$= \frac{\sqrt{x^2-1}}{x} + C$$

(2) Determine if each sequence is convergent or divergent. If convergent, find its limit.

(a) $s_n = \dfrac{2n-1}{n+3}$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n-1}{n+3} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2-\frac{1}{n}}{1+3/n} = \frac{2-0}{1+0} = 2$$

convergent

(b) $a_n = \tan\left(\dfrac{2}{n^2}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan\left(\frac{2}{n^2}\right) = \tan(0) = 0$$

convergent

Note $f(x) = \tan x$ is continuous @ 0

(3) Determine if the series is convergent or divergent. Justify your conclusion.

$$\sum_{n=1}^{\infty} \frac{2n-1}{n+3}$$

Since $\lim\limits_{n\to\infty} \dfrac{2n-1}{n+3} = 2$ from problem 2 a, the series diverges by the divergence test.

(4) Determine if the improper integral is convergent or divergent. If convergent, find its value.

$$\int_0^8 \frac{dx}{\sqrt[3]{x}} = \lim_{t \to 0^+} \int_t^8 x^{-1/3}\, dx$$

$$= \lim_{t \to 0^+} \left. \frac{x^{2/3}}{2/3} \right|_t^8$$

$$= \lim_{t \to 0^+} \left[\frac{3}{2}(8)^{2/3} - \frac{3}{2}(t)^{2/3} \right]$$

$$= \frac{3}{2} \cdot 4 - 0 = 6$$

It's convergent.

(5) Determine if the geometric series is convergent or divergent. If convergent, find its sum.

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^{2n-1}} = \sum_{n=0}^{\infty} \frac{3 \cdot 3^n}{2^{-1} \cdot 2^{2n}} = \sum_{n=0}^{\infty} 6 \left(\frac{3}{4}\right)^n$$

$r = \frac{3}{4}$

so $|r| < 1$

The series converges.

$a = 6$

$$= \frac{6}{1 - \frac{3}{4}}$$

$$= \frac{6}{1 - 3/4} \cdot \frac{4}{4} = \frac{24}{4-3} = 24$$

(6) Determine if the series is absolutely convergent, conditionally convergent or divergent. Clearly justify your conclusion.

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m}$$ Alt. Series Test $a_m = \frac{1}{2m}$

i) $\lim_{m\to\infty} a_m = \lim_{m\to\infty} \frac{1}{2m} = 0$

ii) $a_{m+1} = \frac{1}{2(m+1)} = \frac{1}{2m+2} < \frac{1}{2m} = a_m$

Both conditions hold. Hence the series converges by the alternating series test.

Now, we consider $\sum_{m=1}^{\infty} \left| \frac{(-1)^{m+1}}{2m} \right| = \sum_{m=1}^{\infty} \frac{1}{2m}$

This is $\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m}$, $\frac{1}{2}$ times the divergent harmonic series.

The series $\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m}$ is

conditionally convergent.

(7) Determine if the series is convergent or divergent.

$$\sum_{k=1}^{\infty} \frac{3^k}{5k}$$

Using the ratio test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{3^{k+1}}{5(k+1)} \cdot \frac{5k}{3^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{3\cdot\cancel{3^k}}{\cancel{5}(k+1)} \cdot \frac{\cancel{5}k}{\cancel{3^k}}\right|$$

$$= \lim_{k\to\infty} \frac{3k}{k+1} \cdot \frac{\frac{1}{k}}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{3}{1+\frac{1}{k}} = 3$$

Since $3 > 1$, the series diverges.

(8) Evaluate the indefinite integral. (Hint: The integrand is an **improper** rational function.)

$$\int \frac{x^3 - x^2 + x + 1}{x(x-1)}\, dx$$

Long division:

$$\begin{array}{r|l} & x \\ \hline x^2 - x & x^3 - x^2 + x + 1 \\ & -(x^3 - x^2) \\ \hline & 0 + x + 1 \end{array}$$

so $\dfrac{x^3 - x^2 + x + 1}{x(x-1)} = x + \dfrac{x+1}{x(x-1)}$

partial fractions: $\dfrac{x+1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1} \Rightarrow$

$\Rightarrow \quad x + 1 = A(x-1) + Bx$

$x = 0 \quad 1 = A(-1) \Rightarrow A = -1$

$x = 1 \quad 2 = B(1) \Rightarrow B = 2$

$$\int \frac{x^3 - x^2 + x + 1}{x(x-1)}\, dx = \int \left(x - \frac{1}{x} + \frac{2}{x-1} \right) dx$$

$$= \frac{x^2}{2} - \ln|x| + 2\ln|x-1| + C$$

(9) Determine the form of the partial fraction decomposition of the following rational function. **Note:** It is NOT necessary to find any coefficients A, B, etc.

(a) $\dfrac{x+1}{(x^2+1)(x-4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-4}$

(b) $\dfrac{1}{(x+1)^2(x-2)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-2} + \dfrac{D}{(x-2)^2} + \dfrac{E}{(x-2)^3}$

(10) For each integral or series, choose the one corresponding statement that is true.

(a) The integral $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

(i) is equal to $\ln(x^2) + C$.
(ii) is improper but must converge because the x is squared.
(iii) is improper because it wears white socks with black shoes.
(iv) is improper because the integrand is not continuous on $[0, 1]$.

(b) To evaluate the integral $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

(i) do a partial fraction decomposition on $\frac{x^3}{x^2-1}$, then integrate.
(ii) use a u-substitution with $u = x^2 - 1$.
(iii) do long division followed by a partial fraction decomposition on the proper rational part, then integrate.
(iv) use a trigonometric substitution with $x = \tan\theta$.

(c) Convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3 + 2n - 1}$

(i) can be determined by comparison to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$.
(ii) can be determined by using the alternating series test.
(iii) can be determined by direct comparison to the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$.
(iv) can be determined by the divergence test.

(d) The series of positive terms $\displaystyle\sum_{n=1}^{\infty} a_n$

(i) must be divergent if $\lim_{n\to\infty} a_n = 0$.
(ii) must be convergent if $a_n \le \frac{1}{n^3}$ for each n.
(iii) is a geometric series if $a_n = r^2$ for some real number r.
(iv) is a convergent series if the sequence $\{a_n\}$ is a convergent sequence.