Exam 3 Math 2306 sec. 51

Fall 2015

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas and the provided table of Laplace transforms.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Find the particular solution of the nonhomogeneous equation for which a fundamental solution set to the associated homogeneous equation is provided.

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 2 + \frac{2}{x^2}, \qquad y_1 = x, \quad y_2 = \frac{1}{x}$$

Variation of Parameters

$$g(x) = 2 + \frac{2}{x^2}, \qquad W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{vmatrix} = \frac{-1}{x} - \frac{1}{x} = \frac{-2}{x}$$

$y_p = u_1 y_1 + u_2 y_2$ where

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-\frac{1}{x}\left(2 + \frac{2}{x^2}\right)}{-2/x}\,dx$$

$$= \frac{1}{2}\int \left(2 + \frac{2}{x^2}\right)dx = \int \left(1 + \frac{1}{x^2}\right)dx = x - \frac{1}{x}$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x\left(2 + \frac{2}{x^2}\right)}{-2/x}\,dx$$

$$= \frac{-1}{2}\int x^2\left(2 + \frac{2}{x^2}\right)dx = -\int (x^2 + 1)\,dx = -\frac{x^3}{3} - x$$

$$y_p = \left(x - \frac{1}{x}\right)x - \left(\frac{x^3}{3} + x\right)\cdot\frac{1}{x}$$

$$= x^2 - 1 - \frac{1}{3}x^2 - 1 = \frac{2}{3}x^2 - 2$$

$$\boxed{y_p = \frac{2}{3}x^2 - 2}$$

(2) A 64 lb weight is attached to a spring whose spring constant is 8 lb/ft. The system is undamped and driven by an external force of $F(t) = 2\sin t$. The mass is released from equilibrium with an initial upward velocity of 8 in/sec (2/3 ft/sec). Find the equation of motion. (Take the acceleration due to gravity $g = 32$ ft/sec^2.)

mass $m = \dfrac{64\,\text{lb}}{32\,\frac{\text{ft}}{\text{sec}^2}} = 2$ slugs , $k = 8\,\dfrac{\text{lb}}{\text{ft}}$

$$mx'' + kx = F(t) \Rightarrow 2x'' + 8x = 2\sin t$$

$$x'' + 4x = \sin t \quad , \quad x(0) = 0 \qquad x'(0) = \frac{-2}{3}$$

$$x_c = C_1 \cos 2t + C_2 \sin 2t$$

Using the method of undetermined coefficients, assume

$$x_p = A\sin t + B\cos t$$

$$x_p' = A\cos t - B\sin t$$

$$x_p'' = -A\sin t - B\cos t = -x_p$$

Then $x_p'' + 4x_p = \sin t$

$$3x_p = \sin t \Rightarrow 3A\sin t + 3B\cos t = \sin t$$

$$A = \tfrac{1}{3}, \quad B = 0$$

$$x = C_1\cos 2t + C_2 \sin 2t + \tfrac{1}{3}\sin t$$

$$x'(t) = -2C_1\sin 2t + 2C_2\cos 2t + \tfrac{1}{3}\cos t$$

$x(0) = C_1 = 0$ $\quad x'(0) = 2C_2 + \frac{1}{3} = \frac{-2}{3} \Rightarrow 2C_2 = -1 \quad C_2 = \frac{-1}{2}$

$\boxed{\text{The equation of motion is } x(t) = \frac{-1}{2}\sin 2t + \frac{1}{3}\sin t.}$

(3) Solve the initial value problem using the method of Laplace transforms.

$$y'' + 3y' + 2y = 6e^t, \quad y(0) = 1, \quad y'(0) = -2$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{6e^t\}$$

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 6\mathcal{L}\{e^t\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 3\left(sY(s) - y(0)\right) + 2Y(s) = \frac{6}{s-1}$$

$$(s^2 + 3s + 2)Y(s) - s + 2 - 3 = \frac{6}{s-1}$$

$$(s+1)(s+2)Y(s) = \frac{6}{s-1} + s + 1$$

$$Y(s) = \frac{6}{(s-1)(s+1)(s+2)} + \frac{s+1}{(s+1)(s+2)} = \frac{6}{(s-1)(s+1)(s+2)} + \frac{1}{s+2}$$

Let $\dfrac{6}{(s-1)(s+1)(s+2)} = \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{C}{s+2} \Rightarrow$

$$6 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$

Set $s = 1$ $\quad 6 = 6A \Rightarrow A = 1$

$s = -1$ $\quad 6 = -2B \Rightarrow B = -3$

$s = -2$ $\quad 6 = 3C \Rightarrow C = 2$

$$Y(s) = \frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2} + \frac{1}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{t} - 3e^{-t} + 3e^{-2t}$$

(4) Use any method to evaluate the Laplace transform or inverse transform as indicated.

(a) $\mathscr{L}\{(t-1)^2\}$ $= \mathscr{L}\{t^2 - 2t + 1\}$

$= \mathscr{L}\{t^2\} - 2\mathscr{L}\{t\} + \mathscr{L}\{1\}$

$= \frac{2!}{s^3} - \frac{2}{s^2} + \frac{1}{s}$

(b) $\mathscr{L}^{-1}\left\{\frac{1}{s^6}\right\}$ $= \mathscr{L}^{-1}\left\{\frac{1}{5!} \frac{5!}{s^6}\right\} = \frac{1}{5!} \mathscr{L}^{-1}\left\{\frac{5!}{s^6}\right\} = \frac{1}{5!} t^5$

(c) $\mathscr{L}^{-1}\left\{\frac{s-2}{s^2+4}\right\}$ $= \mathscr{L}^{-1}\left\{\frac{s}{s^2+4} - \frac{2}{s^2+4}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$= \cos 2t - \sin 2t$

(d) $\mathscr{L}\{f'(t)\}$ given $\mathscr{L}\{f(t)\} = \frac{1}{s^{3/2}}$, and $f(0) = 1$

$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$

$= s\left(\frac{1}{s^{3/2}}\right) - 1$

$= \frac{1}{s^{1/2}} - 1$

(5) Use elimination to find the general solution of the system of differential equations.

$$\begin{array}{ccccccc} \frac{dx}{dt} & + & 3x & + & \frac{dy}{dt} & = & 0 \\ 2x & + & \frac{dy}{dt} & - & 10 & = & 0 \end{array}$$

$$\left.\begin{array}{l} (D+3)x + Dy = 0 \\ 2x + Dy = 10 \end{array}\right\} \xRightarrow{\text{subtract}} (D+3)x - 2x = -10$$

$$(D+1)x = -10$$

$x' + x = -10$ 1st order linear w/ integrating factor $\mu = e^{\int dt} = e^t$

$$(e^t x)' = -10e^t$$

$$\int (e^t x)' dt = -\int 10e^t dt = -10e^t + C_1$$

$$x = -10 + C_1 e^{-t}$$

From the 2nd equation $\frac{dy}{dt} = 10 - 2x = 10 - 2(-10 + C_1 e^{-t})$

$$\frac{dy}{dt} = 30 - 2C_1 e^{-t}$$

$$y = \int (30 - 2C_1 e^{-t}) dt = 30t + 2C_1 e^{-t} + C_2$$

The general solution for the system is

$$x = -10 + C_1 e^{-t}$$

$$y = 30t + 2C_1 e^{-t} + C_2$$