Exam 3 Math 2306 sec. 52

Summer 2016

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Find the general solution of the nonhomogeneous equation. The complementary solution is provided.

$$x^2y'' - 4xy' + 6y = -x^2, \quad y_c = c_1x^2 + c_2x^3$$

Standard form $\quad y'' - \frac{4}{x}y' + \frac{6}{x^2}y = -1$

$y_1 = x^2, \quad y_2 = x^3 \qquad W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$

Variation of parameters

$u_1 = -\int \frac{y_2 g}{W}\,dx = -\int \frac{x^3(-1)}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$

$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x^2(-1)}{x^4}\,dx = -\int x^{-2}\,dx = x^{-1}$

$y_p = u_1y_1 + u_2y_2 = \ln x\,(x^2) + x^{-1}\cdot x^3 = x^2\ln x + x^2$

The general solution $\quad y = y_c + y_p$

$y = c_1x^2 + c_2x^3 + x^2\ln x + x^2$

Letting $\quad k_1 = c_1 + 1, \quad k_2 = c_2$

$$\boxed{y = k_1x^2 + k_2x^3 + x^2\ln x}$$

(2) An LRC-series circuit has inductance 1 henry, resistance 2 ohms and capacitance 1 farad. A voltage of $E(t) = 8e^{-3t}$ is applied to the circuit. If the initial charge $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge $q(t)$ on the capacitor for all $t > 0$.

$$Lq'' + Rq' + \frac{1}{C} q = E \quad , \quad L=1, \; R=2, \; C=1, \; E = 8e^{-3t}$$

$$q'' + 2q' + q = 8e^{-3t} \quad , \quad q(0)=0 \quad q'(0)=0$$

Find q_c: $\quad r^2+2r+1=0 \Rightarrow (r+1)^2=0 \Rightarrow r=-1$ repeated

$$q_c = c_1 e^{-t} + c_2 t e^{-t}$$

Find q_p: $\quad$ Let $q_p = Ae^{-3t}$, $\; q_p' = -3Ae^{-3t}$, $\; q_p'' = 9Ae^{-3t}$

$$q_p'' + 2q_p' + q_p = 9Ae^{-3t} - 6Ae^{-3t} + Ae^{-3t} = 8e^{-3t}$$

$$4A = 8 \Rightarrow A = 2$$

The general solution is

$$q = c_1 e^{-t} + c_2 t e^{-t} + 2e^{-3t}$$

$$q' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} - 6e^{-3t}$$

$$q(0) = c_1 + 2 = 0 \Rightarrow c_1 = -2$$

$$q'(0) = -c_1 + c_2 - 6 = 0 \Rightarrow c_2 = 6 + c_1 = 4$$

$$\boxed{\text{The charge } q(t) = -2e^{-t} + 4te^{-t} + 2e^{-3t}.}$$

(3) For each differential equation, determine the **correct form** of the particular solution when using the method of undetermined coefficients. Do not solve for any of the coefficients A, B, etc.

(a) $y''-5y'+6y = 2x+e^{2x}$

$m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0$

$y_1 = e^{2x}$, $y_2 = e^{3x}$

$g_1(x) = 2x$ $\quad y_{p_1} = Ax+B$

$g_2(x) = e^{2x}$ $\quad y_{p_2} = Ce^{2x} \cdot x$

$$\boxed{y_p = Ax+B + Cxe^{2x}}$$

(b) $y''-5y'+6y = x\sin(\pi x)$

Same left side as (a)

$$\boxed{y_p = (Ax+B)\sin(\pi x) + (Cx+D)\cos(\pi x)}$$

(c) $y''+9y = \sin(3x)+e^{3x}$

$m^2 + 9 = 0 \Rightarrow m^2 = -9 \quad m = \pm 3i$

$y_1 = \cos(3x) \quad y_2 = \sin(3x)$

$g_1(x) = \sin(3x)$

$y_{p_1} = (A\sin(3x) + B\cos(3x)) \cdot x$

$g_2(x) = e^{3x}$ $\quad y_{p_2} = Ce^{3x}$

$$\boxed{y_p = Ax\sin(3x) + Bx\cos(3x) + Ce^{3x}}$$

(4) A 1 kg mass is attached to a spring whose spring constant is 16 N/m. A dashpot applies a damping force equivalent to 8 times the instantaneous velocity. No driving force is applied.

(a) Determine if the system is overdamped, underdamped, or critically damped.

$m = 1, \quad \beta = 8, \quad k = 16 \qquad r^2 + 8r + 16 = 0 \Rightarrow (r+4)^2 = 0$

$r = -4$ repeated

The system is critically damped.

(b) If the mass is released from rest (i.e. with zero velocity) from a position 25 cm (i.e. $\frac{1}{4}$ m) below equilibrium, determine the displacement for $t > 0$.

$x(t) = c_1 e^{-4t} + c_2 t e^{-4t} \qquad x(0) = \frac{-1}{4} \quad x'(0) = 0$

$x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$

$x(0) = c_1 = \frac{-1}{4}$

$x'(0) = -4c_1 + c_2 = 0 \Rightarrow c_2 = 4c_1 = -1$

The displacement

$$x(t) = \frac{-1}{4} e^{-4t} - t e^{-4t}$$

(5) Determine the Laplace transform or inverse transform as indicated. Use the table provided and any necessary algebra or function identities.

(a) $\mathscr{L}\left\{(t+3)^2\right\}$ $= \mathscr{L}\{t^2 + 6t + 9\}$

$= \mathscr{L}\{t^2\} + 6\mathscr{L}\{t\} + 9\mathscr{L}\{1\}$

$= \dfrac{2!}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4} + \dfrac{s-2}{s^2+16}\right\}$ $= \dfrac{1}{3!}\mathscr{L}^{-1}\left\{\dfrac{3!}{s^4}\right\} + \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+16}\right\} - \dfrac{2}{4}\mathscr{L}^{-1}\left\{\dfrac{4}{s^2+16}\right\}$

$= \dfrac{1}{3!}t^3 + \cos(4t) - \dfrac{1}{2}\sin(4t)$

(c) $\mathscr{L}\left\{t^3 + 2e^{-t} + 5\right\}$ $= \mathscr{L}\{t^3\} + 2\mathscr{L}\{e^{-t}\} + 5\mathscr{L}\{1\}$

$= \dfrac{3!}{s^4} + \dfrac{2}{s+1} + \dfrac{5}{s}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+4s}\right\}$

$\dfrac{1}{s(s+4)} = \dfrac{A}{s} + \dfrac{B}{s+4}$

$1 = A(s+4) + Bs$

$s=0 \quad 1 = 4A \quad A = \dfrac{1}{4}$

$s=-4 \quad 1 = -4B \quad B = \dfrac{-1}{4}$

$= \dfrac{1}{4}\mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\} - \dfrac{1}{4}\mathscr{L}^{-1}\left\{\dfrac{1}{s+4}\right\}$

$= \dfrac{1}{4} - \dfrac{1}{4}e^{-4t}$

(6) (a) Use the shifting theorem (translation in s) to evaluate the transform or inverse transform as indicated.

(i) $\mathscr{L}\left\{e^{2t}\cos(t)\right\}$

$\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$

$= \frac{s-2}{(s-2)^2+1}$

(ii) $\mathscr{L}^{-1}\left\{\frac{6}{(s-5)^4}\right\}$

$\mathscr{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3$

$= t^3 e^{5t}$

(b) Use the shifting theorem (translation in t) to evaluate the transform or inverse transform as indicated.

(i) $\mathscr{L}\left\{2t\mathscr{U}(t-2)\right\} = e^{-2s}\mathscr{L}\{2(t+2)\} = e^{-2s}\mathscr{L}\{2t+4\}$

$= \frac{2}{s^2}e^{-2s} + \frac{4}{s}e^{-2s}$

(ii) $\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s+1}\right\}$

$\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$= e^{-(t-1)}\mathscr{U}(t-1)$