Exam 3 Math 2306 sec. 53

Fall 2018

Name: (2 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 14 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas. **No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

(1) Evaluate each Laplace transform.

(a) $\mathscr{L}\{(t+1)(t-3)\} = \mathscr{L}\{t^2-2t-3\} = \mathscr{L}\{t^2\} - 2\mathscr{L}\{t\} - 3\mathscr{L}\{1\}$

$= \frac{2}{s^3} - \frac{2}{s^2} - \frac{3}{s}$

(b) $\mathscr{L}\left\{\frac{e^{3t}-e^{-3t}}{2}\right\} = \frac{1}{2}\mathscr{L}\{e^{3t}\} - \frac{1}{2}\mathscr{L}\{e^{-3t}\} = \frac{1}{2}\,\frac{1}{s-3} - \frac{1}{2}\,\frac{1}{s+3}$

(c) $\mathscr{L}\{\sin(2t)-\cos(2t)\} = \frac{2}{s^2+4} - \frac{s}{s^2+4}$

(2) A LC-series circuit (no resistor) with inductance of 10 henries and capacitance of $\frac{1}{10}$ farad has a dying battery attached. The implied electromotive force is $E(t) = 150e^{-2t}$ volts. If the initial charge $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge $q(t)$ on the capacitor for $t > 0$.

$Lq'' + \frac{1}{C} q = E \Rightarrow 10q'' + \frac{1}{1/10} q = 150 e^{-2t}$

Standard form $\quad q'' + q = 15e^{-2t} \quad q(0) = q'(0) = 0$

$q_c: m^2+1=0 \Rightarrow m = \pm i \quad q_c = C_1 \cos(t) + C_2 \sin(t)$

$q_p: q_p = Ae^{-2t} \quad q_p' = -2Ae^{-2t} \quad q_p'' = 4Ae^{-2t}$

$4Ae^{-2t} + Ae^{-2t} = 15e^{-2t} \Rightarrow 5A = 15, \ A = 3$

So the general solution is $q = C_1 \cos t + C_2 \sin t + 3e^{-2t}$

$q' = -C_1 \sin t + C_2 \cos t - 6e^{-2t}$

$q(0) = C_1 + 3 = 0 \quad C_1 = -3$

$q'(0) = C_2 - 6 = 0 \Rightarrow C_2 = 6$

The charge on the capacitor is

$$\boxed{q(t) = -3\cos t + 6\sin t + 3e^{-2t}}$$

(3) Find the general solution of each homogeneous equation.

(a) $\dfrac{d^3y}{dx^3} - 9\dfrac{dy}{dx} = 0$

$m^3 - 9m = 0 \quad m(m-3)(m+3) = 0 \quad m = 0, m = 3, \text{or } m = -3$

$y = C_1 + C_2 e^{3x} + C_3 e^{-3x}$

(b) $4y'' + 4y' + y = 0$

$4m^2 + 4m + 1 = 0 \quad (2m+1) = 0 \quad m = \frac{-1}{2}$ repeated

$y = C_1 e^{-\frac{1}{2}x} + C_2 x e^{-\frac{1}{2}x}$

(4) A 2 kg mass is attached to a spring with spring constant 4 N/m. A dashpot induces damping that is 4 times the instantaneous velocity. In the absence of external driving, set up the homogeneous differential equation describing the displacement $x(t)$ of the mass. Determine if the system is over damped, under damped or critically damped. (It is NOT necessary to solve the ODE.)

$mx'' + \beta x' + kx = 0 \Rightarrow$ $2x'' + 4x' + 4x = 0$ This is the ODE

$r^2 + 2r + 2 = 0$

$r^2 + 2r + 1 + 1 = 0$

$(r+1)^2 = -1$

$r + 1 = \pm i$ Complex roots

$r = -1 \pm i$

The system is underdamped

(5) Find the general solution of the nonhomogeneous equation. The complementary solution is $y_c = c_1x^2 + c_2x^3$.

$$x^2y'' - 4xy' + 6y = -x^2$$

$y'' - \frac{4}{x}y' + \frac{6}{x^2} = -1 \qquad g(x) = -1$

$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \qquad y_p = u_1y_1 + u_2y_2$ where

$u_1 = \int -\frac{g\,y_2}{W}\,dx = \int -\frac{(-1)x^3}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$

$u_2 = \int \frac{g\,y_1}{W}\,dx = \int \frac{-1x^2}{x^4}\,dx = \int \frac{-1}{x^2} = \frac{1}{x}$

$y_p = x^2\ln x + x^3 \cdot \frac{1}{x} = x^2 \ln x + x^2$

we can lump x^2 in with c_1x^2

The general solution is

$y = c_1x^2 + c_2x^3 + x^2\ln x$

(6) Consider the nonhomogeneous equation $y'' + 2y' + y = g(x)$. For each g, determine the **form** of the particular solution when using the method of undetermined coefficients. Do not solve for any coefficients, A, B, etc.

$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \quad m = -1$ repeated

$y_c = c_1 e^{-x} + c_2 x e^{-x}$

(a) $g(x) = 2x + e^{-x}$

$y_{p_1} = Ax + B$

$y_p = Ax + B + Cx^2 e^{-x}$

$y_{p_2} = C e^{-x} \cdot x^2$

(b) $g(x) = x\sin(\pi x)$

$y_p = (Ax+B)\sin(\pi x) + (Cx+D)\cos(\pi x)$

(7) Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^3} + \frac{3}{s+2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= \frac{1}{2!}t^2 + 3e^{-2t}$

(b) $\mathscr{L}^{-1}\left\{\frac{s-1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$

$= \cos(3t) - \frac{1}{3}\sin(3t)$

(c) $\mathscr{L}^{-1}\left\{\frac{6}{s(s-2)}\right\} = -3\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} = -3 + 3e^{2t}$

$\frac{6}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \Rightarrow 6 = A(s-2) + Bs$

$s=0 \quad -2A = 6 \qquad \Rightarrow \quad A = -3$

$s=2 \quad 2B = 6 \qquad\qquad B = 3$