Exam 3 Math 2306 sec. 53 Spring 2019

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Find the general solution of each homogeneous equation.

(a) $y'' - 4y = 0$

$m^2 - 4 = 0$

$m^2 = 4$

$m = 2$ or $m = -2$

$y_1 = e^{2x}, \quad y_2 = e^{-2x}$

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

(b) $y'' - 2y' + 5y = 0$

$m^2 - 2m + 5 = 0$

$(m^2 - 2m + 1) + 4 = 0$

$(m-1)^2 = -4$

$m - 1 = \pm 2i$

$m = 1 \pm 2i$

$y_1 = e^{x}\cos(2x)$

$y_2 = e^{x}\sin(2x)$

$$y = c_1 e^{x}\cos(2x) + c_2 e^{x}\sin(2x)$$

2. A dying battery imparts an electromotive force $E(t) = 10e^{-t}$ to an LC-series circuit (there is no resistor). The inductance is 1 henry and the capacitance is 0.01 (i.e. $\frac{1}{100}$) farads. If the initial charge on the capacitor $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge $q(t)$ for all $t > 0$.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E \qquad \text{Here, } L = 1,\ R = 0,\ C = \frac{1}{100}$$

and $E = 10e^{-t}$

$$q'' + 100q = 10e^{-t}$$

subject to $q(0) = 0 \quad q'(0) = 0$

Find q_c: $m^2 + 100 = 0 \Rightarrow m^2 = -100$

$m = \pm 10i$

So

$$q_c = C_1\cos(10t) + C_2\sin(10t)$$

Find q_p: Using undetermined coefficients,

$$q_p = Ae^{-t},\ q_p' = -Ae^{-t},\ q_p'' = Ae^{-t}$$

$$q_p'' + 100q_p = 10e^{-t}$$

$$Ae^{-t} + 100Ae^{-t} = 10e^{-t} \Rightarrow 101A = 10$$

$$A = \frac{10}{101}$$

So $q(t) = c_1\cos(10t) + c_2\sin(10t) + \frac{10}{101}e^{-t}$

Applying the IC

$$q'(t) = -10c_1\sin(10t) + 10c_2\cos(10t) - \frac{10}{101}e^{-t}$$

$$q(0) = c_1 + \frac{10}{101} = 0 \Rightarrow c_1 = \frac{-10}{101}$$

$$q'(0) = 10c_2 - \frac{10}{101} \Rightarrow c_2 = \frac{1}{10}\left(\frac{10}{101}\right) = \frac{1}{101}$$

The charge

$$\boxed{q(t) = \frac{-10}{101}\cos(10t) + \frac{1}{101}\sin(10t) + \frac{10}{101}e^{-t}}$$

3. Find the general solution of the nonhomogenous equation.

$$y'' + 2y' + y = \frac{e^{-t}}{t}$$

Find y_c: $m^2+2m+1=0$

$(m+1)^2=0 \Rightarrow m=-1$ repeated.

$y_1 = e^{-t}$, $y_2 = te^{-t}$ and $y_c = c_1 e^{-t} + c_2 t e^{-t}$.

Find y_p using variation of parameters.

$$g(t) = \frac{e^{-t}}{t} \qquad W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t}+e^{-t} \end{vmatrix} = -te^{-2t} + e^{-2t} + te^{-2t} = e^{-2t}$$

$$u_1 = \int \frac{-g y_2}{W}\,dt = \int \frac{-\frac{e^{-t}}{t} \cdot te^{-t}}{e^{-2t}}\,dt = \int 1\,dt = t$$

$$u_2 = \int \frac{g y_1}{W}\,dt = \int \frac{\frac{e^{-t}}{t}\, e^{-t}}{e^{-2t}}\,dt = \int \frac{1}{t}\,dt = \ln t$$

Then $y_p = u_1 y_1 + u_2 y_2 = te^{-t} + t\ln t\, e^{-t}$

The te^{-t} term can be combined w/ y_c.

The general solution

$$\boxed{y = c_1 e^{-t} + c_2 t e^{-t} + t \ln t\, e^{-t}}$$

4. Suppose you wish to use the method of undetermined coefficients to find a particular solution to the differential equation

$$y'' + y' - 6y = g(x).$$

For each function $g(x)$, determine the form of the particular solution. (Do not solve for any of the coefficients A, B, etc.)

y_c: $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3,\ m = 2$

The complementary solution is

$y_c = C_1 e^{-3x} + C_2 e^{2x}$

(a) $g(x) = 4xe^{2x} - 2\cos(2x)$

$y_{p_1} = (Ax+B)e^{2x} \cdot x$

$y_{p_2} = C\cos(2x) + D \cdot \sin(2x)$

$y_p = (Ax^2 + Bx)e^{2x} + C\cos(2x) + D\sin(2x)$

(b) $g(x) = \dfrac{e^{3x}}{2} + \dfrac{e^{-3x}}{2}$

$y_{p_1} = Ae^{3x}$

$y_{p_2} = Be^{-3x} \cdot x$

$y_p = Ae^{3x} + Bxe^{-3x}$

(c) $g(x) = 4xe^{x}\cos(\pi x)$

$y_p = (Ax+B)e^{x}\cos(\pi x) + (Cx+D)e^{x}\sin(\pi x)$

5. Evaluate each Laplace transform or inverse Laplace transform.

(a) $\mathscr{L}\left\{t^3 + 4\cos(\pi t)\right\}$ $= \mathscr{L}\{t^3\} + 4\mathscr{L}\{\cos(\pi t)\}$

$$= \frac{3!}{s^4} + \frac{4s}{s^2+\pi^2}$$

(b) $\mathscr{L}\left\{(e^{2t}+1)^2\right\}$ $= \mathscr{L}\{e^{4t} + 2e^{2t} + 1\}$

$$= \frac{1}{s-4} + \frac{2}{s-2} + \frac{1}{s}$$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{s^4} + \frac{1}{s^2+9}\right\}$ $= \mathscr{L}^{-1}\left\{\frac{1}{3!}\ \frac{3!}{s^4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{3}\ \frac{3}{s^2+3^2}\right\}$

$$= \frac{1}{3!}t^3 + \frac{1}{3}\sin(3t)$$

(d) $\mathscr{L}^{-1}\left\{\frac{3s}{(s-1)(s+2)}\right\}$ $= \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{t} + 2e^{-2t}$

$$\frac{3s}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} \Rightarrow 3s = A(s+2) + B(s-1)$$

$s=1$ $\quad 3s = 3A \quad A = 1$

$s=-2$ $\quad -6 = -3B \quad B = 2$