Exam 3 Math 2306 sec. 54

Fall 2015

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 65 problems worth 20 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas and the provided table of Laplace Transforms.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) A 64 lb weight is attached to a spring whose spring constant is 128 lb/ft. The surrounding medium offers a damping force that is numerically equal to 32 times the instantaneous velociy. The mass is released from rest (i.e. zero velocity) from a position 1 ft above equilibrium. Determine the equation of motion.

mass $\quad m = \dfrac{64 \text{ lb.}}{32 \ \frac{ft}{sec^2}} = 2$ slugs $\qquad k = 128 \ \frac{lb}{ft}, \quad \beta = 32$

$$mx'' + \beta x' + kx = 0, \qquad x(0) = -1 \text{ ft} \qquad x'(0) = 0$$

$$2x'' + 32x' + 128x = 0 \qquad x(0) = -1, \ x'(0) = 0$$

$$x'' + 16x' + 64x = 0$$

Characteristic eqn: $\quad r^2 + 16r + 64 = 0$

$(r+8)^2 = 0 \Rightarrow r = -8$ repeated

$$x = C_1 e^{-8t} + C_2 t e^{-8t}$$

$$x'(t) = -8C_1 e^{-8t} - 8C_2 t e^{-8} + C_2 e^{-8t}$$

$$x(0) = C_1 = -1, \quad x'(0) = -8C_1 + C_2 = 0 \Rightarrow C_2 = 8C_1 = -8$$

Finally, the equation of motion is

$$x = -e^{-8t} - 8te^{-8t}$$

(2) Find the particular solution of the nonhomogeneous equation for which a fundamental solution set to the associated homogeneous equation is provided.

$$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = x^{-1/2}, \qquad y_1 = \frac{1}{x}, \quad y_2 = x^3$$

$g(x) = x^{-1/2}$, $W = \begin{vmatrix} \frac{1}{x} & x^3 \\ \frac{-1}{x^2} & 3x^2 \end{vmatrix} = 3x + x = 4x$

With Variation of parameter $y_p = u_1 y_1 + u_2 y_2$

where

$$u_1 = -\int \frac{y_2 g}{W} dx = -\int \frac{x^3 x^{-1/2}}{4x} dx = \frac{-1}{4}\int x^{3/2} dx = \frac{-1}{4} \frac{x^{5/2}}{5/2} = \frac{-1}{10} x^{5/2}$$

$$u_2 = \int \frac{y_1 g}{W} dx = \int \frac{\frac{1}{x} x^{-1/2}}{4x} dx = \frac{1}{4}\int x^{-5/2} dx = \frac{1}{4} \frac{x^{-3/2}}{-3/2} = \frac{-1}{6} x^{-3/2}$$

$$y_p = \frac{-1}{10} x^{5/2} \cdot \frac{1}{x} - \frac{1}{6} x^{-3/2} \cdot x^3 = \frac{-1}{10} x^{3/2} - \frac{1}{6} x^{3/2}$$

$$= \left(\frac{-3-5}{30}\right) x^{3/2} = \frac{-4}{15} x^{3/2}$$

The particular solution

$$y_p = \frac{-4}{15} x^{3/2}$$

(3) Solve the initial value problem using the method of Laplace Transforms.

$$y'' + y' - 2y = 12e^{-3t}, \quad y(0) = 1, \quad y'(0) = 1$$

Let $\mathcal{L}\{y\} = Y(s)$

$$\mathcal{L}\{y'' + y' - 2y\} = \mathcal{L}\{12e^{-3t}\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 12\,\mathcal{L}\{e^{-3t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) + (sY(s) - y(0)) - 2Y(s) = \frac{12}{s+3}$$

$$(s^2 + s - 2)Y(s) - s - 2 = \frac{12}{s+3}$$

$$(s+2)(s-1)Y(s) = \frac{12}{s+3} + s + 2$$

$$Y(s) = \frac{12}{(s+3)(s+2)(s-1)} + \frac{s+2}{(s+2)(s-1)}$$

Decompose $\displaystyle \frac{12}{(s+3)(s+2)(s-1)} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{s-1}$

$$12 = A(s+2)(s-1) + B(s+3)(s-1) + C(s+3)(s+2)$$

Set $s = -3$ $\quad 12 = 4A \Rightarrow A = 3$

$s = -2$ $\quad 12 = -3B \Rightarrow B = -4$

$s = 1$ $\quad 12 = 12C \Rightarrow C = 1$

$$Y(s) = \frac{3}{s+3} - \frac{4}{s+2} + \frac{1}{s-1} + \frac{1}{s-1}$$

$$y = \mathcal{L}^{-1}\{Y(s)\} = 3e^{-3t} - 4e^{-2t} + 2e^{t}$$

(4) Use elimination to find the general solution of the system of differential equations.

$$\begin{array}{ccccccc} \frac{dx}{dt} & + & 2x & + & \frac{dy}{dt} & = & 0 \\ 3x & + & \frac{dy}{dt} & - & t & = & 0 \end{array}$$

$$\left.\begin{array}{r} (D+2)x + Dy = 0 \\ 3x + Dy = t \end{array}\right\} \xRightarrow{\text{subtract}} \begin{array}{l} (D+2)x - 3x = -t \\ (D-1)x = -t \end{array}$$

$x' - x = -t$, 1st order linear w/ integrating factor $\mu = e^{-\int dt} = e^{-t}$

$(e^{-t}x)' = -te^{-t} \Rightarrow \int (e^{-t}x)' dt = -\int te^{-t} dt$

$u = t \quad du = dt$

$v = -e^{-t} \quad dv = e^{-t} dt$

$e^{-t} x = te^{-t} - \int e^{-t} dt$

$= te^{-t} + e^{-t} + C_1$

$x = t + 1 + C_1 e^{t}$

From the 2nd equation, $\frac{dy}{dt} = t - 3x = t - 3(t + 1 + C_1 e^{t})$

$= -2t - 3 - 3C_1 e^{t}$

$y = \int (-2t - 3 - 3C_1 e^{t}) dt = -t^2 - 3t - 3C_1 e^{t} + C_2$

The system has general solution

$x = C_1 e^{t} + t + 1$

$y = -t^2 - 3t - 3C_1 e^{t} + C_2$

(5) Use any method to evaluate the Laplace transform or inverse transform as indicated.

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{4!}\ \frac{4!}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\}$

$= \frac{1}{4!}t^4$

(b) $\mathscr{L}\left\{(t+2)^2\right\} = \mathscr{L}\left\{t^2 + 4t + 4\right\}$

$= \mathscr{L}\{t^2\} + 4\mathscr{L}\{t\} + 4\mathscr{L}\{1\}$

$= \frac{2!}{s^3} + \frac{4}{s^2} + \frac{4}{s}$

(c) $\mathscr{L}^{-1}\left\{\frac{s-3}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} - \frac{3}{s^2+9}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= \cos 3t - \sin 3t$

(d) $\mathscr{L}\left\{f'(t)\right\}$ given $\mathscr{L}\left\{f(t)\right\} = \frac{2}{s^{5/2}}$, and $f(0) = -1$

$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$

$= s\left(\frac{2}{s^{5/2}}\right) - (-1)$

$= \frac{2}{s^{3/2}} + 1$