Exam 3 Math 2306 sec. 54 Spring 2019

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Find the general solution of each homogeneous equation.

(a) $y'' - 25y = 0$

$m^2 - 25 = 0$

$m^2 = 25$

$m = \pm 5$

$y = C_1 e^{5x} + C_2 e^{-5x}$

(b) $y'' - 4y' + 5y = 0$

$m^2 - 4m + 5 = 0$

$m^2 - 4m + 4 + 1 = 0$

$(m-2)^2 = -1$

$m - 2 = \pm i$

$m = 2 \pm i$

$y = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x$

2. A dying battery imparts an electromotive force $E(t) = 20e^{-t}$ to an LC-series circuit (there is no resistor). The inductance is 2 henry and the capacitance is 0.005 (i.e. $\frac{1}{200}$) farads. If the initial charge on the capacitor $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge $q(t)$ for all $t > 0$.

$L = 2$, $C = \frac{1}{200}$, $E = 20e^{-t}$ and $R = 0$

$Lq'' + Rq' + \frac{1}{C}q = E$

$2q'' + 200q = 20e^{-t}$

In standard form $q'' + 100q = 10e^{-t}$

$m^2 + 100 = 0 \Rightarrow m = \pm 10i$

So $q_c = C_1 \cos(10t) + C_2 \sin(10t)$

Using undetermined coefficients, set $q_p = Ae^{-t}$

Then $q_p' = -Ae^{-t}$, $q_p'' = Ae^{-t}$

$Ae^{-t} + 100Ae^{-t} = 10e^{-t} \Rightarrow 101A = 10$

$A = \frac{10}{101}$

So $q_p = \frac{10}{101}e^{-t}$

$q = C_1 \cos(10t) + C_2 \sin(10t) + \frac{10}{101}e^{-t}$

$q' = -10C_1 \sin(10t) + 10C_2 \cos(10t) - \frac{10}{101}e^{-t}$

$q(0) = C_1 + \frac{10}{101} = 0 \Rightarrow C_1 = \frac{-10}{101}$

$q'(0) = 10C_2 - \frac{10}{101} \Rightarrow C_2 = \frac{1}{10}\left(\frac{10}{101}\right) = \frac{1}{101}$

The charge

$$q(t) = \frac{-10}{101}\cos(10t) + \frac{1}{101}\sin(10t) + \frac{10}{101}e^{-t}$$

for $t > 0$.

3. Find the general solution of the nonhomogenous equation.

$$y'' + 2y' + y = \frac{9e^{-t}}{\sqrt{t}}$$

Find y_c: $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0 \Rightarrow m = -1$ repeated

$y_1 = e^{-t}$, $y_2 = te^{-t}$

Using variation of parameters $y_p = u_1 y_1 + u_2 y_2$

Here $g(t) = \frac{9}{\sqrt{t}} e^{-t}$ and $W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix}$

$= e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$

$$u_1 = \int \frac{-g y_2}{W} dt = \int \frac{-\frac{9}{\sqrt{t}} e^{-t}(te^{-t})}{e^{-2t}} dt = -9 \int \sqrt{t}\, dt$$

$$= -9 \frac{t^{3/2}}{3/2} = -6t^{3/2}$$

$$u_2 = \int \frac{g y_1}{W} dt = \int \frac{\frac{9}{\sqrt{t}} e^{-t}(e^{-t})}{e^{-2t}} dt = 9 \int t^{-1/2} dt = 9 \frac{t^{1/2}}{1/2}$$

$$= 18 t^{1/2}$$

$$y_p = -6t^{3/2} e^{-t} + 18 t^{1/2} t e^{-t} = 12 t^{3/2} e^{-t}$$

The solution

$$\boxed{y = c_1 e^{-t} + c_2 t e^{-t} + 12 t^{3/2} e^{-t}}$$

4. Suppose you wish to use the method of undetermined coefficients to find a particular solution to the differential equation

$$y'' + y' - 12y = g(x).$$

For each function $g(x)$, determine the form of the particular solution. (Do not solve for any of the coefficients A, B, etc.)

Consider y_c

$m^2 + m - 12 = 0$

$(m+4)(m-3) = 0$

$m = -4$ or $m = 3$

$y_c = C_1 e^{-4x} + C_2 e^{3x}$

(a) $g(x) = 4xe^{3x} - 2\cos(2x)$

$g_1 \quad g_2$

$y_{p_1} = (Ax+B)e^{3x} \cdot x$

$= (Ax^2+Bx)e^{3x}$

$y_{p_2} = C\cos(2x) + D\sin(2x)$

$y_p = (Ax^2+Bx)e^{3x} + C\cos(2x) + D\sin(2x)$

(b) $g(x) = 4xe^{x}\cos(\pi x)$

$y_p = (Ax+B)e^{x}\cos(\pi x) + (Cx+D)e^{x}\sin(\pi x)$

5. Evaluate each Laplace transform or inverse Laplace transform.

(a) $\mathscr{L}\{3t^4 - 2\sin(6t)\} = 3\mathscr{L}\{t^4\} - 2\mathscr{L}\{\sin(6t)\}$

$$= 3\frac{4!}{s^5} - 2\frac{6}{s^2+36} = \frac{72}{s^5} - \frac{12}{s^2+36}$$

(b) $\mathscr{L}\{(1-e^{-t})^2\} = \mathscr{L}\{1 - 2e^{-t} + e^{-2t}\}$

$$= \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{s^3} + \frac{3}{s^2+25}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2!}\frac{2!}{s^3}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{5}\frac{5}{s^2+5^2}\right\}$

$$= \frac{1}{2}t^2 + \frac{3}{5}\sin(5t)$$

(d) $\mathscr{L}^{-1}\left\{\frac{-2s}{(s-1)(s-3)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$$= e^{t} - 3e^{3t}$$

$$\frac{-2s}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3} \Rightarrow$$

$-2s = A(s-3) + B(s-1)$

$s=1 \quad -2 = -2A \quad A = 1$

$s=3 \quad -6 = 2B \quad B = -3$