Exam 3 Math 2306 sec. 56
Fall 2017

Name: ____________________________ Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________________

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INSTRUCTIONS: There are 7 problems; the point values are listed with the problems. You may use one sheet (8.5” × 11”) of your own prepared notes/formulas.

No use of a calculator, textbook, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.
(1) (10 points) Find the general solution of the nonhomogeneous equation. The complementary solution is provided.

\[ x^2 y'' - 4xy' + 6y = -x^2, \quad y_c = c_1 x^2 + c_2 x^3 \]

\[ y'' - \frac{4}{x} y' + \frac{6}{x^2} y = -1 \quad y_1 = x^2, \quad y_2 = x^3 \]

\[ w = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^7 - 2x^3 = x^4 \]

\[ y_p = u_1 y_1 + u_2 y_2 \]

\[ u_1 = \int -\frac{y_2 w}{w} \, dx = \int -\frac{(x^3)}{x^4} \, dx = \int \frac{1}{x} \, dx = \ln x \]

\[ u_2 = \int \frac{y_1 w}{w} \, dx = \int \frac{x^2}{x^4} \, dx = \int x^2 \, dx = -\left( \frac{x^3}{3} \right) = -\frac{1}{x} \]

\[ y_p = x^2 \ln x + \frac{1}{3} (x^3) = x^2 \ln x + x^2 \]

we can combine the two with \( c_1 x^2 \).

\[ y = c_1 x^2 + c_2 x^3 + x^2 \ln x \]
(2) (15 points) An LRC-series circuit has inductance 1 henry, resistance 2 ohms and capacitance 1 farad. A voltage of \( E(t) = 8e^{-3t} \) is applied to the circuit. If the initial charge \( q(0) = 0 \) and the initial current \( i(0) = 0 \), determine the charge \( q(t) \) on the capacitor for all \( t > 0 \).

\[
Lq'' + Rq' + \frac{1}{C} q = E \quad q(0) = 0 \quad q'(0) = 0
\]

\[
q'' + 2q' + q = 8e^{-3t}
\]

For \( q_c \) \( \quad r^2 + 2r + 1 = 0 \quad (r + 1)^2 = 0 \quad r = -1 \quad \text{repeated} \)

\[
q_c = C_1 e^t + C_2 te^t
\]

For \( q_p \)

\[
p \quad q_p = Ae^{-3t} \quad q_p' = -3Ae^{-3t} \quad q_p'' = 9Ae^{-3t}
\]

\[
9Ae^{-3t} - 6Ae^{-3t} + Ae^{-3t} = 8e^{-3t}
\]

\[
4A = 8 \quad A = 2
\]

\[
q_p = 2e^{-3t}
\]

\[
q = q_c + q_p = C_1 e^t + C_2 te^t + 2e^{-3t}
\]

\[
q' = C_1 e^t + C_2 e^t - C_2 e^{-3t} - 6e^{-3t}
\]

\[
q(0) = C_1 + 2 = 0 \quad \Rightarrow \quad C_1 = -2
\]

\[
q'(0) = -C_1 + C_2 - 6 = 0 \quad \Rightarrow \quad C_2 = 6 + C_1 = 4
\]

\[
q(t) = -2e^t + 4te^t + 2e^{-3t}
\]
(3) (15 points, 5 each) For each differential equation, determine the correct form of the particular solution when using the method of undetermined coefficients. Do not solve for any of the coefficients $A$, $B$, etc.

(a) $y'' - 5y' + 6y = 2x + e^{2x}$

$m^2 - 5m + 6 = 0 \quad (m-2)(m-3) = 0$

$y_c = c_1 e^{2x} + c_2 e^{3x}$

$y_p = \text{Ax} + \text{B}$

$y_p = \text{Cx} e^{2x}$

$y_p = \text{Ax} + \text{B} + \text{Cx} e^{2x}$

(b) $y'' - 5y' + 6y = x \sin(\pi x)$

Some $y_c \text{ or (a)}$

$y_p = (\text{Ax} + \text{B}) \sin(\pi x) + (\text{Cx} + \text{D}) \cos(\pi x)$

(c) $y'' + 9y = \sin(3x) + e^{3x}$

$m^2 + 9 = 0 \quad m = \pm 3i$

$y_c = c_1 \cos(3x) + c_2 \sin(3x)$

$y_p_1 = \text{Ax} \sin(3x) + \text{Bx} \cos(3x)$

$y_p_2 = \text{Ce}^{3x}$

$y_p = \text{Ax} \sin(3x) + \text{Bx} \cos(3x) + \text{Ce}^{3x}$
(4) A 1 kg mass is attached to a spring whose spring constant is 16 N/m. A dashpot applies a damping force equivalent to 8 times the instantaneous velocity. No driving force is applied.

(a) (5 points) Determine if the system is overdamped, underdamped, or critically damped.

\[ mx'' + \beta x' + kx = 0 \quad x'' + 8x' + 16x = 0 \]

\[ \lambda^2 + 8\lambda + 16 = 0 \]

\[ (\lambda + 4)^2 = 0 \]

\[ \lambda = -4 \quad \text{repeated} \]

Critical damping

(b) (10 points) If the mass is released from rest (i.e. with zero velocity) from a position 25 cm (i.e. \( \frac{1}{4} \) m) below equilibrium, determine the displacement for \( t > 0 \).

\[ x = c_1 e^{-4t} + c_2 te^{-4t} \]

\[ x(0) = \frac{1}{4} \quad x'(0) = 0 \]

\[ x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t} \]

\[ x(0) = c_1 = \frac{1}{4} \]

\[ x'(0) = -4c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = 4c_1 = 4 \left( \frac{1}{4} \right) = 1 \]

\[ x = \frac{1}{4} e^{-4t} - t e^{-4t} \]
(5) (20 points, 5 each) Determine the Laplace transform or inverse transform as indicated. Use the table provided and any necessary algebra or function identities.

(a) \[ \mathcal{L}\{(t+3)^2\} = \mathcal{L}\{t^2+6t+9\} \]
\[ = \mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} + 9\mathcal{L}\{1\} \]
\[ = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \]

(b) \[ \mathcal{L}^{-1}\left\{\frac{1}{s^4} + \frac{s-2}{s^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3} + \frac{2i}{s^2} + \frac{s-2}{s^2+16}\right\} \]
\[ = \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{2i}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+16}\right\} \]
\[ = \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{2i}{s^2}\right\} + \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{y}{s^2+16}\right\} \]
\[ = \frac{1}{6} t^3 + \cos(4t) - \frac{1}{2} \sin(4t) \]

(c) \[ \mathcal{L}\{t^3 + 2e^{-t} + 5\} = \mathcal{L}\{t^3\} + 2\mathcal{L}\{e^{-t}\} + 5\mathcal{L}\{1\} \]
\[ = \frac{3!}{s^4} + \frac{2}{s+1} + \frac{5}{s} \]

(d) \[ \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\} = \frac{A}{s} + \frac{B}{s+4} \Rightarrow 1 = A(s+4) + Bs \]
\[ s=0 \quad A = \frac{1}{4} \]
\[ s=-4 \quad B = \frac{1}{4} \]
\[ \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = \frac{1}{t} - \frac{1}{4} e^{-4t} \]
(6) (20 points, 5 each) For each nonhomogeneous equation, determine whether the method of undetermined coefficients could be used to find a particular solution \( y_p \). If not, briefly state why the method could not be used.

(a) \( y'' - 2 \sin xy + 3y = \sin x \)

\[ \text{No, not constant coefficient} \]

(b) \( 4y'' + 4y' + y = e^x \cos(\pi x) \)

\[ \text{Yes} \]

(c) \( y'' - 3y' + 14y = x \tan x \)

\[ \text{No, tan on right side is not the right function type} \]

(d) \( x^2y'' + 2xy' - 3y = x^2 + 2x - 3 \)

\[ \text{No, not constant coefficient} \]

(7) (5 points) Note that if \( f(t) = t \) and \( g(t) = t^2 \), then \( f(t)g(t) = t^3 \). Use these functions and their Laplace transforms to demonstrate that

\[ \mathcal{L} \{ f(t)g(t) \} \neq \mathcal{L} \{ f(t) \} \mathcal{L} \{ g(t) \} \]

\[ \mathcal{L} \{ t \} = \frac{1}{s^2}, \quad \mathcal{L} \{ t^2 \} = \frac{2}{s^3} \]

\[ \mathcal{L} \{ t^3 \} = \frac{6}{s^4} + \frac{1}{s^2} \cdot \frac{2}{s^3} = \frac{2}{s^5} \]