Exam 3 Math 2306 sec. 57

Fall 2017

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems; the point values are listed with the problems. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) (10 points) Find the general solution of the nonhomogeneous equation. The complementary solution is provided.

$$x^2y'' - 4xy' + 6y = x^3, \quad y_c = c_1x^2 + c_2x^3$$

$y'' - \frac{4}{x} y' + \frac{6}{x^2} y = x$ $\qquad$ $y_1 = x^2$, $y_2 = x^3$

$g(x) = x$ $\qquad$ $W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$

$y_p = u_1 y_1 + u_2 y_2$

$u_1 = \int \frac{-g y_2}{W} dx = \int \frac{-x x^3}{x^4} dx = \int -1 \, dx = -x$

$u_2 = \int \frac{g y_1}{W} dx = \int \frac{x \cdot x^2}{x^4} dx = \int \frac{1}{x} dx = \ln x$

$y_p = -x \cdot x^2 + x^3 \ln x = -x^3 + x^3 \ln x$

we can combine $-x^3$ with c_2x^3

$$\boxed{y = c_1 x^2 + c_2 x^3 + x^3 \ln x}$$

(2) (15 points) An LRC-series circuit has inductance 1 henry, resistance 2 ohms and capacitance 1 farad. A voltage of $E(t) = 27e^{-4t}$ is applied to the circuit. If the initial charge $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge $q(t)$ on the capacitor for all $t > 0$.

$$Lq'' + Rq' + \frac{1}{C} q = E$$

$$q'' + 2q' + q = 27e^{-4t} \qquad q(0) = q'(0) = 0$$

$$r^2 + 2r + 1 = 0 \qquad (r+1)^2 = 0 \Rightarrow r = -1 \text{ repeated}$$

$$q_c = C_1 e^{-t} + C_2 t e^{-t}$$

For q_p, put $q_p = Ae^{-4t}$, $q_p' = -4Ae^{-4t}$, $q_p'' = 16Ae^{-4t}$

$$16Ae^{-4t} - 8Ae^{-4t} + Ae^{-4t} = 27e^{-4t}$$

$$9A = 27 \qquad A = 3$$

$$q = C_1 e^{-t} + C_2 t e^{-t} + 3e^{-4t}$$

$$q' = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t} - 12e^{-4t}$$

$$q(0) = C_1 + 3 = 0 \Rightarrow C_1 = -3$$

$$q'(0) = -C_1 + C_2 - 12 = 0 \Rightarrow C_2 = 12 + C_1 = 9$$

$$\boxed{q = -3e^{-t} + 9te^{-t} + 3e^{-4t}}$$

(3) (15 points, 5 each) For each differential equation, determine the **correct form** of the particular solution when using the method of undetermined coefficients. Do not solve for any of the coefficients A, B, etc.

(a) $y''-7y'+12y = 3x+e^{3x}$

$m^2 - 7m + 12 = 0$ $\quad (m-3)(m-4) = 0$

$y_c = C_1 e^{3x} + C_2 e^{4x}$

$y_{p_1} = Ax + B$

$y_{p_2} = Cx\,e^{3x}$

$$\boxed{y_p = Ax + B + Cx\,e^{3x}}$$

(b) $y''-7y'+12y = x\sin(\pi x)$

Same y_c as part (a)

$$\boxed{y_p = (Ax+B)\sin(\pi x) + (Cx+D)\cos(\pi x)}$$

(c) $y''+25y = \sin(5x)+e^{5x}$

$m^2 + 25 = 0 \Rightarrow m = \pm 5i$ $\quad y_c = C_1\cos(5x) + C_2\sin(5x)$

$y_{p_1} = Ax\sin(5x) + Bx\cos(5x)$

$y_{p_2} = C e^{5x}$

$$\boxed{y_p = Ax\sin(5x) + Bx\cos(5x) + Ce^{5x}}$$

(4) A 1 kg mass is attached to a spring whose spring constant is 25 N/m. A dashpot applies a damping force equivalent to 10 times the instantaneous velocity. No driving force is applied.

(a) (5 points) Determine if the system is overdamped, underdamped, or critically damped.

$mx'' + \beta x' + kx = 0$ $\quad\quad x'' + 10x' + 25x = 0$

$r^2 + 10r + 25 = 0$

$(r+5)^2 = 0$

$r = -5$ repeated

Critically Damped

(b) (10 points) If the mass is released from rest (i.e. with zero velocity) from a position 20 cm (i.e. $\frac{1}{5}$ m) above equilibrium, determine the displacement for $t > 0$.

$x = c_1 e^{-5t} + c_2 t e^{-5t}$ $\quad\quad x(0) = \frac{1}{5}, \quad x'(0) = 0$

$x' = -5c_1 e^{-5t} + c_2 e^{-5t} - 5c_2 t e^{-5t}$

$x(0) = c_1 = \frac{1}{5}$

$x'(0) = -5c_1 + c_2 = 0$ $\quad\quad c_2 = 5c_1 = 5 \cdot \frac{1}{5} = 1$

$$\boxed{x = \frac{1}{5} e^{-5t} + t e^{-5t}}$$

(5) (20 points, 5 each) Determine the Laplace transform or inverse transform as indicated. Use the table provided and any necessary algebra or function identities.

(a) $\mathscr{L}\left\{t^3 + 2e^{-t} + 5\right\}$ $= \mathscr{L}\{t^3\} + 2\mathscr{L}\{e^{-t}\} + 5\,\mathscr{L}\{1\}$

$= \dfrac{3!}{s^4} + \dfrac{2}{s+1} + \dfrac{5}{s}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^5} + \dfrac{s+1}{s^2+25}\right\}$ $= \mathscr{L}^{-1}\left\{\dfrac{1}{4!}\,\dfrac{4!}{s^5} + \dfrac{s}{s^2+5^2} + \dfrac{1}{5}\,\dfrac{5}{s^2+5^2}\right\}$

$= \dfrac{1}{4!}\,\mathscr{L}^{-1}\left\{\dfrac{4!}{s^5}\right\} + \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+5^2}\right\} + \dfrac{1}{5}\,\mathscr{L}^{-1}\left\{\dfrac{5}{s^2+5^2}\right\}$

$= \dfrac{1}{4!}\,t^4 + \cos(5t) + \dfrac{1}{5}\sin(5t)$

(c) $\mathscr{L}\left\{(t-3)^2\right\}$ $= \mathscr{L}\{t^2 - 6t + 9\}$

$= \mathscr{L}\{t^2\} - 6\,\mathscr{L}\{t\} + 9\,\mathscr{L}\{1\}$

$= \dfrac{2!}{s^3} - \dfrac{6}{s^2} + \dfrac{9}{s}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+3s}\right\}$

$\dfrac{1}{s(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+3} \Rightarrow 1 = A(s+3) + Bs$

$s=0 \quad A = \frac{1}{3}$

$s=-3 \quad B = \frac{-1}{3}$

$= \mathscr{L}^{-1}\left\{\dfrac{\frac{1}{3}}{s} - \dfrac{\frac{1}{3}}{s+3}\right\}$

$= \dfrac{1}{3}\,\mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\} - \dfrac{1}{3}\,\mathscr{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

$= \dfrac{1}{3} - \dfrac{1}{3}e^{-3t}$

(6) (20 points, 5 each) For each nonhomogeneous equation, determine whether the method of undetermined coefficients could be used to find a particular solution y_p. **If not, briefly state why the method could not be used.**

(a) $y''-2y+3y = x\ln(3x)$

No, log on the right is not the right kind of function

(b) $4y''+4y'+y = x^3e^x$

yes

(c) $y''-3y'+14y = \sec(\pi x)$

No, secant function is of the wrong type

(d) $x^2y''+4xy'+5y = x^2+4x+5e^{x}$

No, not constant coefficient

(7) (5 points) Note that if $f(t) = t$ and $g(t) = t^3$, then $f(t)g(t) = t^4$. Use these functions and their Laplace transforms to demonstrate that

$$\mathscr{L}\{f(t)g(t)\} \neq \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2}, \quad \mathcal{L}\{g(t)\} = \frac{3!}{s^4}$$

$$\mathcal{L}\{t^4\} = \frac{4!}{s^5} \neq \frac{1}{s^2}\cdot\frac{3!}{s^4} = \frac{6}{s^6}$$