Exam 4 Math 2254H sec. 015H

Spring 2015

Name: 2 points Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 14 points each. No use of notes, books, or calculator is allowed. Illicit use of notes, a book, calculator, or any smart device will result in a grade of zero on this exam and may result in a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Determine if the given series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=0}^{\infty} a_n \quad \text{where} \quad a_0 = -2, \quad \text{and} \quad a_{n+1} = \left(\frac{2n+1}{n+3}\right) a_n$$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2n+1}{n+3} \right|$$

$$= \lim_{n\to\infty} \left(\frac{2n+1}{n+3}\right) \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{1+3/n} = 2$$

$L = 2 > 1$

The series diverges by the ratio test.

(2) Although the following series is not truly a p-series, it is comparable to a p-series. Find the value of p for the p-series which can be used for a comparison test. Based on this value of p, is the given series expected to **converge** or to **diverge**?

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{\sqrt[3]{n^7+2n^3-1}}$$

$p =$ 4/3

as $n \to \infty$

$$\frac{\sqrt{n^2+1}}{\sqrt[3]{n^7+2n^3-1}} \sim \frac{\sqrt{n^2}}{\sqrt[3]{n^7}} = \frac{n}{n^{7/3}} = \frac{1}{n^{4/3}}$$

The p-series w/ $p = \frac{4}{3}$ is convergent.

This series is expected to Convergent.

(3) Determine if the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$$

Alternating series test: $b_n = \frac{1}{\sqrt{n}-1}$

i) $b_{n+1} = \frac{1}{\sqrt{n+1}-1} < \frac{1}{\sqrt{n}-1} = b_n$ for all $n \geq 2$

so $b_{n+1} \leq b_n$

ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$

This series converges by the alternating series test.

(4) Use the ratio test to show that the given series is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^3}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3} = \frac{\cancel{(n!)^3}(n+1)^3\,(3n)!}{(3n+3)!\,\cancel{(n!)^3}}$$

$$= \frac{(n+1)^3\,(3n)!}{(3n)!\,(3n+1)(3n+2)(3n+3)} = \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)}$$

So

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3}{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)} = \frac{1^3}{3^3} = \frac{1}{27}$$

Since $L = \frac{1}{27} < 1$ the series is absolutely convergent.

(5) Based on your results in problem (4), find the radius of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{(n!)^3 x^n}{(3n)!}$$

Here $a_n = \dfrac{(n!)^3 x^n}{(3n)!}$. Using the above,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3\,|x|}{(3n+1)(3n+2)(3n+3)} = \frac{|x|}{27}$$

$$\frac{|x|}{27} < 1 \implies |x| < 27.$$

The radius $R = 27$.

(6) Select the true statement for each of (a), (b), and (c)

(a) The series $\sum_{n=0}^{\infty} \frac{5^{2n}}{3^{3n-1}}$

$5^{2n} = (5^2)^n = 25^n$

$3^{3n-1} = 3^{3n} \cdot \frac{1}{3} = \frac{1}{3}(3^3)^n = \frac{1}{3} 27^n$

i) is geometric with common ratio $r = \frac{5}{3}$.

ii) is a divergent p-series with $p = \frac{2}{3}$.

iii) is geometric with common ratio $r = \frac{25}{27}$.

iv) is geometric with first term $a = \frac{1}{3}$ and common ratio $r = \frac{25}{9}$.

(b) The series $\sum_{n=0}^{\infty} (-1)^n b_n$

i) is an alternating series for any sequence of numbers $\{b_n\}$. only if $b_n \geq 0 \ \forall n$

ii) is convergent if $\lim_{n\to\infty} b_n = 0$.

iii) is convergent if the series $\sum_{n=0}^{\infty} |b_n|$ is convergent. (Def. of absolute convergence)

iv) is the alternating harmonic series.

(c) The series $\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^5}$

i) is a p-series with $p = 5$.

ii) is convergent by the ratio test.

iii) is a geometric series with first term $a = 4$ and common ratio $r = \frac{\sqrt{n}}{n^5}$.

iv) is a convergent p-series.

(7) Find a power series representation centered at zero for the function f and identify its radius of convergence. (Remember that $\tan^{-1}(0) = 0$.)

$f(x) = \tan^{-1}(2x)$ $\left(\text{Hint:} \quad f'(x) = \dfrac{2}{1+4x^2}\right)$

$$f'(x) = \frac{2}{1+4x^2} = \frac{2}{1-(-4x^2)} = \sum_{n=0}^{\infty} 2(-4x^2)^n$$

$$= \sum_{n=0}^{\infty} 2(-1)^n 4^n x^{2n}$$

If $|-4x^2| < 1 \Rightarrow 4|x^2| < 1$, $|x^2| < \frac{1}{4} \Rightarrow |x| < \frac{1}{2}$

$$f(x) = \int f'(x)\,dx = C + \sum_{n=0}^{\infty}\left(2(-1)^n 4^n \int x^{2n}\,dx\right) \quad |x| < \tfrac{1}{2}$$

$$= C + \sum_{n=0}^{\infty} 2(-1)^n 4^n \frac{x^{2n+1}}{2n+1}$$

$$f(x) = C + 2x - 8\frac{x^3}{3} + \dots$$

$$f(0) = 0 = C + 0 - 0 + \dots \Rightarrow C = 0$$

So $$f(x) = \sum_{n=0}^{\infty} \frac{2(-1)^n 4^n x^{2n+1}}{2n+1} \quad \text{for } |x| < \tfrac{1}{2}$$

The radius of convergence $R = \frac{1}{2}$.