

Exam 4 Math 2306 sec. 51

Fall 2015

Name: _____ *Solutions* _____

Your signature (required) confirms that you agree to practice academic honesty.

Signature: _____

| Problem | Points |
|---------|--------|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" \times 11") of your own prepared notes/formulas and the provided table of Laplace transforms.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Use any method to evaluate the Laplace transform or inverse transform as indicated.

$$(a) \quad \mathcal{L}\{(t-3)^2 \mathcal{U}(t-3)\} = e^{-3s} \mathcal{L}\{t^2\} = \frac{2e^{-3s}}{s^3}$$

$$(b) \quad \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s-2}\right\} = e^{2(t-4)} \mathcal{U}(t-4)$$

$$\begin{aligned} (c) \quad \mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\} \\ &= e^{2t} \cos 2t + e^{2t} \sin 2t \end{aligned}$$

$$(d) \quad \mathcal{L}\{e^{-\pi t} t^3\} = \frac{3!}{(s+\pi)^4}$$

(2) Find the Fourier series of f .

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \leq x < 1 \end{cases}$$

here, $P=1$

$$a_0 = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_0^1 x dx = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2}$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(x) \cos(n\pi x) dx = \int_0^1 x \cos(n\pi x) dx$$

$$u = x \quad du = dx \\ dv = \cos(n\pi x) dx$$

$$v = \frac{1}{n\pi} \sin(n\pi x)$$

$$= \left. \frac{x}{n\pi} \sin(n\pi x) \right|_0^1 - \frac{1}{n\pi} \int_0^1 \sin(n\pi x) dx$$

$$= \frac{1}{n^2 \pi^2} \cos(n\pi x) \Big|_0^1 = \frac{1}{n^2 \pi^2} (\cos(n\pi) - \cos(0)) = \frac{(-1)^n - 1}{n^2 \pi^2}$$

$$b_n = \frac{1}{1} \int_{-1}^1 f(x) \sin(n\pi x) dx = \int_0^1 x \sin(n\pi x) dx$$

$$u = x \quad du = dx$$

$$dv = \sin(n\pi x) dx$$

$$v = -\frac{1}{n\pi} \cos(n\pi x)$$

$$= \left. -\frac{x}{n\pi} \cos(n\pi x) \right|_0^1 + \frac{1}{n\pi} \int_0^1 \cos(n\pi x) dx$$

$$= -\frac{1}{n\pi} \cos(n\pi) - 0 + \frac{1}{n^2 \pi^2} \sin(n\pi x) \Big|_0^1$$

$$= -\frac{1}{n\pi} (-1)^n$$

$$= \frac{(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi^2} \cos(n\pi x) + \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

(3) Solve the initial value problem using the method of Laplace transforms.

$$y' + 2y = 4te^{-2t}, \quad y(0) = -3$$

$$\text{Let } \mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4te^{-2t}\}$$

$$sY(s) - y(0) + 2Y(s) = 4 \frac{1}{(s+2)^2}$$

$$(s+2)Y(s) = \frac{4}{(s+2)^2} - 3$$

$$Y(s) = \frac{4}{(s+2)^3} - \frac{3}{s+2}$$

$$\text{Note } \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} = 2 \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = 2t^2$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= 2t^2 e^{-2t} - 3e^{-2t}$$

(4) An LR series circuit with inductance 1 h and resistance 2 ohms is attached to a battery with an open switch. After 1 second, the switch is closed applying a constant 40 volts. Find the current if the initial current is $i(0) = 0$ A. That is, solve the IVP

$$\frac{di}{dt} + 2i = \begin{cases} 0, & 0 \leq t < 1 \\ 40, & t \geq 1 \end{cases} \quad i(0) = 0$$

$$\text{let } \mathcal{L}\{i\} = I(s)$$

$$= 40u(t-1)$$

$$\mathcal{L}\{i' + 2i\} = \mathcal{L}\{40u(t-1)\}$$

$$sI(s) - i(0) + 2I(s) = \frac{40e^{-s}}{s}$$

$$(s+2)I(s) = \frac{40e^{-s}}{s} \Rightarrow I(s) = \frac{40e^{-s}}{s(s+2)}$$

$$\text{Decompose: } \frac{40}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \Rightarrow 40 = A(s+2) + Bs$$

$$\text{let } s=0 \quad 40 = 2A \Rightarrow A = 20$$

$$s=-2 \quad 40 = -2B \Rightarrow B = -20$$

$$I(s) = \frac{20e^{-s}}{s} - \frac{20e^{-s}}{s+2}$$

$$i(t) = \mathcal{L}^{-1}\{I(s)\}$$

$$= 20u(t-1) - 20e^{-2(t-1)}u(t-1)$$

(5) (a) Find the half range sine series of f . Then, (b) on the graph provided, plot three periods over the interval $(-3\pi, 3\pi)$ of the sine series found in part (a).

$$f(x) = -1, \quad 0 < x < \pi$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} -1 \sin(nx) dx = \frac{2}{n\pi} \cos(nx) \Big|_0^{\pi}$$

$$= \frac{2}{n\pi} (\cos(n\pi) - \cos 0) = \frac{2}{n\pi} ((-1)^n - 1)$$

$$\text{a) } f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} ((-1)^n - 1) \sin(nx)$$

