Exam 4 Math 2306 sec. 53

Fall 2018

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas. **No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

(1) Evaluate each Laplace transform.

(a) $\mathscr{L}\{e^{-\pi t}\cos(2t)\}$ $= \dfrac{s+\pi}{(s+\pi)^2+4}$

(b) $\mathscr{L}\left\{(1+e^{2t})\mathscr{U}(t-3)\right\}$ $= e^{-3s}\mathscr{L}\{1+e^{2(t+3)}\}$

$= e^{-3s}\left(\dfrac{1}{s}+\dfrac{e^6}{s-2}\right)$

(c) Given that : $\mathscr{L}\{t\sin t\} = \dfrac{2s}{(s^2+1)^2}$, evaluate $\mathscr{L}\{te^{-t}\sin t\}$ $= \dfrac{2(s+1)}{\left((s+1)^2+1\right)^2}$

(2) Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{e^{-6s}}{s^4}\right\} = \frac{1}{3!}(t-6)^3 u(t-6)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}t^3$

(b) $\mathscr{L}^{-1}\left\{\frac{s}{s^2-4s+8}\right\} = \mathscr{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+2^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s-2}{(s-2)^2+2^2}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{(s-2)^2+2^2}\right\}$

$= e^{2t}\cos(2t) + e^{2t}\sin(2t)$

$s^2-4s+8 = s^2-4s+4+4$
$= (s-2)^2+2^2$

(c) $\mathscr{L}^{-1}\left\{\frac{e^{-\frac{\pi}{3}s}}{s^2+9}\right\} = \frac{1}{3}\sin\left(3\left(t-\frac{\pi}{3}\right)\right)u(t-\pi/3)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$

(3) Use the method of Laplace transforms to solve the first order initial value problem.

$\frac{dy}{dx} - 2y = 6, \quad y(0) = 3$ $\qquad \mathscr{L}\{y'-2y\} = \mathscr{L}\{6\}$ $\qquad$ let $Y = \mathscr{L}\{y\}$

$sY(s) - y(0) - 2Y(s) = \frac{6}{s}$

$(s-2)Y(s) = \frac{6}{s} + 3 \quad \Rightarrow \quad Y(s) = \frac{6}{s(s-2)} + \frac{3}{s-2}$

$\frac{6}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$

$6 = A(s-2) + Bs$

$s=0 \quad 6=-2A \quad A=-3$

$s=2 \quad 6=2B \quad B=3$

$Y(s) = \frac{-3}{s} + \frac{3}{s-2} + \frac{3}{s-2}$

$= \frac{-3}{s} + \frac{6}{s-2}$

$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -3 + 6e^{2t}$

(4) Find the Fourier series for $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$. $p = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = \frac{x}{\pi}\Big|_0^{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx$$

$$= \frac{1}{n\pi}\sin(nx)\Big|_0^{\pi} = 0 \quad \text{for all } n \geq 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= \frac{-1}{n\pi}\cos(nx)\Big|_0^{\pi} = \frac{-1}{n\pi}\left(\cos(n\pi) - \cos(0)\right)$$

$$= \frac{-1}{n\pi}\left((-1)^n - 1\right) = \frac{1-(-1)^n}{n\pi}$$

Finally

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin(nx)$$

(5) Note that $e^{3t} = e^t \times e^{2t}$. Use $f(t) = e^t$ and $g(t) = e^{2t}$ to demonstrate that

$$\mathscr{L}\{f(t)g(t)\} \neq \mathscr{L}\{f(t)\} \times \mathscr{L}\{g(t)\}$$

$$\mathscr{L}\{f(t)\} = \frac{1}{s-1} \quad \text{and} \quad \mathscr{L}\{g(t)\} = \frac{1}{s-2}$$

$$\mathscr{L}\{f(t)\,g(t)\} = \mathscr{L}\{e^{3t}\} = \frac{1}{s-3} \neq \frac{1}{(s-1)(s-2)}$$

i.e. $\mathscr{L}\{f(t)g(t)\}$ is <u>NOT</u> $\mathscr{L}\{f(t)\} \cdot \mathscr{L}\{g(t)\}$

(6) A 1 kg mass is attached to a spring with spring constant 16 N/m. No damping is applied, but a 16 N external stretching force is applied for 2 seconds and then released. If the mass starts from rest at equilibrium, determine the displacement of the mass for all time $t > 0$. That is, solve the following initial value problem. Use the method of Laplace transforms.

$$x'' + 16x = 16 - 16\mathscr{U}(t-2), \qquad x(0) = 0, \quad x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$\mathcal{L}\{x'' + 16x\} = \mathcal{L}\{16 - 16\mathcal{U}(t-2)\}$$

$$s^2 X - s\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + 16X(s) = \frac{16}{s} - \frac{16}{s}e^{-2s}$$

$$(s^2+16)X(s) = \frac{16}{s} - \frac{16}{s}e^{-2s}$$

$$X(s) = \frac{16}{s(s^2+16)} - \frac{16\,e^{-2s}}{s(s^2+16)}$$

Decomp: $$\frac{16}{s(s^2+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+16}$$

$$16 = A(s^2+16) + (Bs+C)s$$

$$= (A+B)s^2 + Cs + 16A$$

$$\left.\begin{aligned} A+B &= 0 \\ C &= 0 \\ 16A &= 16 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned}$$

$$X(s) = \frac{1}{s} - \frac{s}{s^2+16} - \left(\frac{1}{s} - \frac{s}{s^2+16}\right)e^{-2s}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$\boxed{x(t) = 1 - \cos(4t) - \left(1 - \cos(4(t-2))\right)\mathcal{U}(t-2)}$$