Exam 4 Math 2306 sec. 53 Spring 2019

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Evaluate each Laplace transform.

(a) $\mathscr{L}\left\{t^3e^{-3t}\right\} = \dfrac{3!}{(s+3)^4}$

(b) $\mathscr{L}\left\{t^2\mathscr{U}(t-1)\right\} = e^{-s}\mathscr{L}\{(t+1)^2\} = e^{-s}\mathscr{L}\{t^2+2t+1\}$

$= e^{-s}\left(\dfrac{2}{s^3}+\dfrac{2}{s^2}+\dfrac{1}{s}\right)$

2. Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2-4s+5}\right\}$ $= \mathcal{L}^{-1}\left\{\dfrac{s-2}{(s-2)^2+1}\right\} + 2\,\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^2+1}\right\}$

$$= e^{2t}\cos t + 2e^{2t}\sin t$$

$s^2-4s+5 = (s-2)^2+1$

and

$s = s-2+2$

(b) $\mathscr{L}^{-1}\left\{\dfrac{e^{-\frac{\pi}{2}s}}{s^2+4}\right\}$ $= \frac{1}{2}\sin\left(2(t-\pi/2)\right)\mathcal{U}(t-\frac{\pi}{2})$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{2}\sin(2t)$$

3. Solve the first order initial value problem using the method of Laplace transforms.

$\frac{dy}{dt} + 2y = 4 - te^{-2t}, \quad y(0) = 1$ Let $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4 - te^{-2t}\}$

$sY - y(0) + 2Y = \frac{4}{s} - \frac{1}{(s+2)^2}$

$(s+2)Y - 1 = \frac{4}{s} - \frac{1}{(s+2)^2}$

$Y(s) = \frac{4}{s(s+2)} - \frac{1}{(s+2)^3} + \frac{1}{s+2}$

$\frac{4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \Rightarrow 4 = A(s+2) + Bs$

$s=0 \quad 2A = 4 \quad A = 2$

$s=-2 \quad -2B = 4 \quad B = -2$

so $Y(s) = \frac{2}{s} - \frac{2}{s+2} - \frac{1}{(s+2)^3} + \frac{1}{s+2}$

$= \frac{2}{s} - \frac{1}{s+2} - \frac{1}{(s+2)^3}$

using $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{1}{2}t^2$

$y(t) = 2 - e^{-2t} - \frac{1}{2}t^2e^{-2t}$

4. Use the method of Laplace transforms to solve the initial value problem.

$$y'' + y = \begin{cases} 0, & 0 \le t < \pi \\ 3, & t \ge \pi \end{cases} \qquad y(0) = 0, \quad y'(0) = 1$$

$= 3u(t-\pi)$

$\mathcal{L}\{y'' + y\} = \mathcal{L}\{3u(t-\pi)\}$

$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{3}{s} e^{-\pi s}$

$(s^2+1)Y(s) - 1 = \frac{3}{s} e^{-\pi s}$

$Y(s) = \frac{3}{s(s^2+1)} e^{-\pi s} + \frac{1}{s^2+1}$

$\frac{3}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 3 = A(s^2+1) + (Bs+C)s$

$= (A+B)s^2 + Cs + A$

$A+B=0$

$C=0 \Rightarrow B = -A = -3$

$A=3$

$Y(s) = \frac{3}{s} e^{-\pi s} - \frac{3s}{s^2+1} e^{-\pi s} + \frac{1}{s^2+1}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$y(t) = 3u(t-\pi) - 3\cos(t-\pi)u(t-\pi) + \sin t$

5. Suppose that $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 2, & 0 < x < \pi \end{cases}$

(a) Determine the Fourier series representation of f.

(b) Sketch a graph of the Fourier series (what it converges to) over the interval $(-3\pi, 3\pi)$.

(c) What value does the series converge to when $x = \frac{\pi}{2}$? When $x = \pi$?

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} dx + \int_0^{\pi} 2\,dx\right] = \frac{1}{\pi}\left[x\Big|_{-\pi}^{0} + 2x\Big)\Big|_0^{\pi}\right] = \frac{1}{\pi}[3\pi] = 3$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} 2\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{\frac{1}{n}\sin(nx)\Big|_{-\pi}^{0}}_{0} + \underbrace{\frac{2}{n}\sin(nx)\Big|_0^{\pi}}_{0}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} 2\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\cos(nx)\Big|_{-\pi}^{0} + \frac{-2}{n}\cos(nx)\Big|_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left[\frac{-1}{n} - \frac{-1}{n}(-1)^n + \frac{-2}{n}(-1)^n - \frac{-2}{n}\right] = \frac{1}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]$$

$$= \frac{1-(-1)^n}{n\pi}$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin(nx)$$

@ $\frac{\pi}{2}$, the series converges to 2

@ π, it converges to $\frac{3}{2}$